Generalized fixed points on strictly convex hypersurfaces

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Abstract

In 1992 Claude Viterbo used generating functions to define symplectic capacities for compactly supported Hamiltonian diffeomorphisms φ of the special symplectic manifold $\mathbb{R}^{2n}$. He applied them to define capacities for open sets in $\mathbb{R}^{2n}$. We extend the definition of these capacities to arbitrary sets in $\mathbb{R}^{2n}$ and prove that this generalization is non trivial: the capacity of a strictly convex hypersurface Σ is equal to the capacity of an open bounded set U with $\partial U = \Sigma$ which is a positive number.

We prove this by considering a Hamiltonian function which is constant on U and zero on a neighbourhood of U and deform it to a Hamiltonian function with support in a neighbourhood of Σ. Via the Hamiltonian equations we can associate to these functions Hamiltonian isotopies. We show that the deformation can be performed without changing the capacities of the time one diffeomorphisms. To do this we explicitely compute the Maslov indices of periodic orbits of the flow. We use a definition of the Maslov index from Claude Viterbo and David Theret and develop an algorithm to compute the indices via the spectral flow of a finite dimensional operator.

We then apply this result to show a generalized fixed point theorem which is a partial generalization of a result of Jürgen Moser from 1978. Moser considered any compact simply connected coisotropic submanifold A of an exact symplectic manifold M and an exact symplectic diffeomorphism φ which is C^1–close to the identity. He proved that there exist two points $x \in A$ such that $\varphi(x)$ and x are on the same leaf of the characteristic foliation on A.

We only consider strictly convex hypersurfaces $\Sigma \subset \mathbb{R}^{2n}$. Under the hypothesis that the capacity of φ is smaller than the capacity of U we show that there exists a point $x \in \Sigma$ which is mapped under φ onto its own leaf.

This answers a question of Helmut Hofer who in 1989 proved a similiar result formulated in terms of Hofer's displacement energy and capacities defined by Ivar Ekeland and Helmut Hofer. His result is valid for hypersurfaces of restricted contact type. He asked whether one could use Viterbo's capacities instead to find bounds on the capacities of symplectic diffeomorphisms that would guarantee the existence of a point which is mapped onto its own leaf.

Zusammenfassung

Claude Viterbo benutzte 1992 Erzeugende Funktionen, um symplektische Kapazitäten für Hamiltonsche Diffeomorphismen mit kompaktem Träger im $\mathbb{R}^{2n}$ zu definieren. Damit definierte er dann Kapazitäten für offene Mengen im $\mathbb{R}^{2n}$. Wir erweitern die Definition dieser Kapazitäten auf beliebige Mengen im $\mathbb{R}^{2n}$ und zeigen, dass die Definition nicht trivial ist: Die Kapazität einer strikt konvexen Hyperfläche Σ ist gleich der Kapazität einer beschränkten offenen Menge U mit $\partial U = \Sigma$. Kapazitäten offener Mengen sind positiv.

Den Beweis führen wir, indem wir eine Hamiltonsche Funktion betrachten, die konstant auf U ist und Träger in einer Umgebung von U hat. Wir deformieren diese in eine Funktion mit Träger in einer Umgebung von Σ. Über die Hamiltonschen Differentialgleichungen erhalten wir dann Hamiltonsche Isotopien. Wir zeigen, dass die Deformation ausgeführt werden kann, ohne die Kapazitäten der Zeit–1–Diffeomorphismen zu ändern. Dabei berechnen wir explizit die Maslov Indizes der periodischen Orbits des Flusses. Wir benutzen eine Definition des Maslov Index, die auf Claude Viterbo und David Theret zurückgeht und entwickeln einen Algorithmus, den Maslov Index über den Spektralfluss eines endlichdimensionalen Operators auszurechnen.

Diese Resultate wenden wir an, um einen verallgemeinerten symplektischen Fixpunktsatz zu zeigen, der eine teilweise Verallgemeinerung eines Resultats von Jürgen Moser aus dem Jahre 1978 ist. Moser betrachtete eine beliebige kompakte, einfach zusammenhängende, coisotrope Untermannigfaltigkeit A einer exakt symplektischen Mannigfaltigkeit M und einen exakt symplektischen Diffeomorphismus φ, der C^1–nahe bei der Identität ist. Er zeigte, dass zwei Punkte $x \in A$ existieren, so dass $\varphi(x)$ und x auf dem gleichen Blatt der charakteristischen Blätterung auf A liegen.

Wir betrachten nur strikt konvexe Hyperflächen $\Sigma \subset \mathbb{R}^{2n}$. Unter der Voraussetzung, dass die Kapazität von φ kleiner als die Kapazität von Σ ist, zeigen wir, dass ein Punkt $x \in \Sigma$ existiert, der unter φ auf sein eigenes Blatt abgebildet wird.

Das beantwortet eine Frage von Helmut Hofer. Dieser benutzte 1989 seine 'displacement' Energie und Kapazitäten, die von Ivar Ekeland und Helmut Hofer definiert wurden, um ein ähnliches Resultat für Hyperflächen vom eingeschränkten Kontakt–Typ zu zeigen. Hofer fragte, ob Viterbos Kapazitäten genutzt werden können, um Schranken für Kapazitäten symplektischer Diffeomorphismen zu finden, so dass die Existenz eines Punktes garantiert ist, der auf sein eigenes Blatt abgebildet wird.

Periodic hobbits

In [To] J.R.R. Tolkien stated that

In a hole in the ground there lived a hobbit. Not a nasty, dirty, wet hole, filled with the ends of worms and an oozy smell, nor yet a dry, bare, sandy hole with nothing in it to sit down on or to eat: it was a hobbit-hole, and that means comfort.

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Contents

1 Introduction

Generalized fixed points

Many problems in symplectic geometry lead to questions like whether Lagrangian submanifolds of a symplectic manifold (M, ω) intersect or whether a symplectic diffeomorphim φ has fixed points. One can connect these two questions if one looks at the diagonal Δ in $(M \times M, (-\omega) \oplus \omega)$, and intersects Δ with the graph of φ. The intersection points of these two Lagrangians are the fixed points of φ.

In 1978 J. Moser presented in his paper 'A Fixed Point Theorem in Symplectic Geometry', [M], another point of view to bring these two problems together. He considered a compact simply connected coisotropic (see appendix A in this work) submanifold $A \subset M$. It is foliated by k–dimensional leaves if the codimension of A is k. Denote the leaf through $x \in A$ by $L_A(x)$. Given a symplectic diffeomorphism φ which is close to the identity Moser proved (see theorem 2.1) that there are at least two points which are mapped onto its own leaf: $\varphi(x) \in L_A(x)$. This can be considered as a 'generalized fixed point'.

For a Lagrangian submanifold A the leaves are n–dimensional, hence $L_A(x) = A$. Consequently $\varphi(A) \cap A$ consists of at least two points. The other extreme is that $A = M$. In this case the leaves are points and one considers symplectic fixed points.

There are interesting intermediate cases, for example hypersurfaces $\Sigma \subset M$ which are automatically coisotropic. The leaves are one dimensional. Moser then considered in the case $n > 1$ the harmonic oscillator given by the Hamiltonian function $H_0(q,p) = \frac{1}{2}\Sigma a_i(q_i^2 + p_i^2)$ and a function $H_1 : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ with support in $[0,1] \times \mathbb{R}^{2n}$. Denote by φ_t the Hamiltonian flow associated to H_1. Moser proved that for every $c > 0$ there exists a point $x \in H^{-1}(c) =: \Sigma$ such that $\varphi(x) \in L_\Sigma(x)$. This is valid in the case $n = 1$ as well. To see this we tell a little story:

Suppose little Jana[1] *is sitting on a swing. There is no friction so her energy is conserved. Then there comes a short (Hamiltonian) breeze slowing her down*[2]*. She looses energy. She thinks that if only the same breeze had hit her at another point in her movement it would have accelerated her.*

[1] see http://www.math.ethz.ch/~linne/jana.jpg

[2] ok, that is not consistent — now there is friction — every model has its disadvantages

So there has to be a point in her movement where the same breeze would leave her on her old energy level again. If the breeze is too stormy it would have accelerated her in any case — which could be very uncomfortable.

This little example shows that the natural question that arises is the following: *Can the smallness condition in Moser's theorem be replaced by geometric conditions on Σ and φ?*

This work deals with the above question. Before explaining our approach via 'generating functions' we give another partial answer which was derived by H. Hofer in his paper 'On the topological properties of symplectic maps', [H], by infinite dimensional variational methods.

Capacities

Hofer used symplectic invariants, the symplectic capacities (see definition 2.2). The first capacity, the so–called 'symplectic width' was discovered by M. Gromov in his famous paper 'Pseudo–Holomorphic Curves in Almost Complex Manifolds', [Gr]. The capacity c_{EH} that Hofer used was defined by E. Ekeland and H. Hofer in 'Symplectic Topology and Hamiltonian Dynamics', [EH]. Hofer then defined the energy of symplectic diffeomorphisms with compact support[3].

He considered a hypersurface of restricted contact type (see definition A.1) with capacity $c_{EH}(\Sigma)$ and a symplectic diffeomorphism φ with energy $E(\varphi) \leq c_{EH}(\Sigma)$. He proved that there exists a point $x \in \Sigma$ such that x and $\varphi(x)$ are on the same leaf of the characteristic foliation on Σ.

see theorem 2.5.

Generating functions

In the search for fixed points and Lagrangian intersections there have been used in the past years two main techniques to tackle problems in symplectic geometry. In 1978 P. Rabinowitz, [R], discovered that the degenerate action functional of classical mechanics can be successfully used for existence results. Afterwards, A. Floer, H. Hofer and many others used infinite dimensional action functionals on the loop space of any symplectic manifolds to prove strong existence results.

[3]This energy can be used to define a bi–invariant metric on the space of symplectic diffeomorphisms with compact support.

The other approach is used in the framework of cotangent bundles of closed manifolds. It stays in finite dimensions and uses generating functions, a concept which was already known to H. Poincaré in the last century. L. Hörmander in [Ho] extended the definition of generating functions to higher dimensional vector bundles to solve problems in partial differential equations.

Generating functions describe Lagrangians in the cotangent bundle of a given manifold B. One question is: *Which Lagrangians admit generating functions?*

In his paper 'Une idée du type géodésiques brisés pour les systèmes hamiltoniens', [Ch], M. Chaperon in 1984 presented an idea how to construct generating functions for graphs by a broken geodesic method. The idea was prompted by the Lyapunov–Schmidt reduction method of the classical action functional in the proof of the Arnold concecture of the torus.

In 1986 J. C. Sikorav utilized in his paper 'Sur les immersions lagrangiennes admettant une phase génératrice globale', [S], this idea to prove for cotangent bundles that the property of having a generating function is invariant under Hamiltonian isotopy. He even proved that the generating function can be chosen to be quadratic at infinity, see theorem 3.4.

In his paper 'Symplectic Topology as the Geometry of Generating Functions', [V2] C. Viterbo in 1992 showed that the 'generating functions quadratic at infinity' constructed this way are unique up to some natural operations, see theorem 3.5[4].

With the help of the uniqueness theorem C. Viterbo was able to construct symplectic capacities for open sets in $\mathbb{R}^{2n}$ and also for symplectic diffeomorphisms.

It is not clear how far Ekeland/Hofer's and Viterbo's capacities agree. H. Hofer then asked whether Viterbo's capacities can be used to prove a result similar to his theorem mentioned above.

In our work we answer this question positively for the case of strictly convex hypersurfaces.

[4]There were some imprecisements in his proof, namely an incorrect reference to a theorem of J. Cerf, [Ce]. The proof was cleared in the thesis of D. Theret, [Th].

We next present a summary of each chapter of this thesis. (Chapter 1 is this introduction.)

Chapter 2

It is devoted to the formulation of the theorems of Hofer and Moser. Furthermore, most of the basic definitions are stated.

Chapter 3

The third section deals with the concept of generating functions. Sikorav's existence and Viterbo's uniqueness theorem can be found here. We recall the definition of Viterbo's capacities and some of their properties from Viterbo's paper [V2], adding proofs where Viterbo has skipped them. For proposition 3.8, S. Born in his diploma thesis has presented an improved proof which we add as well. Furthermore some inequalities we need in the last section are shown (Proposition 3.13).

In addition, we extend Viterbo–capacities to arbitrary subsets of $\mathbb{R}^{2n}$. We prove that the Viterbo–capacity of the unit sphere is π, thus showing that Viterbo's capacities are non trivial, even for sets which are not open.

Chapter 4

In the fourth section we introduce a version of the Maslov index which is adapted to generating functions. There are several possibilities to generalize the Maslov index for Lagrangian loops (see [A]) and the related Conley–Zehnder index for periodic orbits (see [CZ]) to the case of arbitrary paths. Viterbo in [V1] introduced a generating function version of this index proving that it really is the Maslov index by comparing it to the index in [CZ] and [Du]. Theret in [Th] proved that this index satisfies the axioms of a Maslov index as defined by Capell/Lee/Miller in [CLM] without leaving the context of generating functions.

Out of Theret's work we derive a generic formula (proposition 4.7) for the index which nicely reflects the 'broken geodesic' nature of generating functions quadratic at infinity: In the construction of generating functions for Hamiltonian isotopies one writes the isotopy as a composition of symplectic maps

$$\varphi_1 = (\varphi_1 \circ \varphi_{t_n}^{-1}) \circ \ldots \circ (\varphi_{t_i} \circ \varphi_{t_{i-1}}^{-1}) \circ \ldots \circ (\varphi_{t_1} \circ I).$$

Here the 'breaking points' t_k are chosen such that $\ker(d(\varphi_t \circ \varphi_{t_{i-1}}^{-1})(x) + I) = \{0\}$ for all $t \in]t_{i-1}, t_i]$ and all $x \in \mathbb{R}^{2n}$.

Our formula for the Maslov index states that one has to compute the spectral flow of a finite–dimensional symmetric operator — but only as long as $d\varphi_t$ has no eigenvalue -1. In the generic case defined in proposition 4.7 there is only a finite number of times t_k such that $d\varphi_t$ has eigenvalue -1.

Thus we compute the indices on time intervals $]t_{k-1}, t_k]$ and then add up these indices.

In contrast, the Maslov index defined intrinsically by H. Hofer, C. Wysocki and E. Zehnder in [HWZ1] and [HWZ2] uses the spectral flow of an infinite dimensional self adjoint operator — and one does not need to break up.

Thus in the context of generating functions one stays in the finite dimensional realm — for the prize of having to break up and getting more complicated formulas.

We then prove that this formula is really generic: Every path of symplectic matrices can be deformed without changing its Maslov index to one to which the formula applies.

Chapter 5

In this section we prove that for a strictly convex closed hypersurface Σ and a bounded set U with $\partial U = \Sigma$ the capacities agree:

$$c(U) = c(\Sigma).$$

To do this we have to deform a Hamiltonian function H which is constant on U and 0 outside a small neighbourhood of U to a function which is constant on a small neighbourhood of Σ and 0 on a slightly bigger neighbourhood without changing the capacities of the associated Hamiltonian diffeomorphisms.

In the proof we need to control the Maslov indices of the periodic orbits during the deformation — which we can do with the help of the generic formula.

Chapter 6

We consider a strictly convex closed hypersurface with capacity $c(\Sigma)$ and a Hamiltonian diffeomorphism φ satisfying $c(\varphi) < c(\Sigma)$. We prove that there always exists a point $x \in \Sigma$ such that x and $\varphi(x)$ are on the same leaf of the characteristic foliation on Σ.

We next sketch the proof. For U with $\partial U = \Sigma$ we can vary the size of U by multiplying every $x \in U$ with a real number α, thus obtaining αU.

As in chapter 4 we consider a sequence of Hamiltonians H_k with support in $(1 + 1/k)U \setminus (1 - 1/k)U$ such that the capacities of the time one flows are close to $c(\Sigma)$.

These Hamiltonians have for $|\delta| < 1/k$ the strictly convex hypersurfaces $(1 + \delta)\Sigma$ as regular energy surfaces. We then find a point $x \in (1 + \delta)\Sigma$ such that its image $\varphi(x)$ is on the same leaf:

$$\varphi(x_k) \in L_{(1+\delta)\Sigma}(x_k).$$

We find a bound on the length of the leaves independent of k. Passing to the limit $k \to \infty$ and applying Arzela–Ascoli yields that there is a point $x \in \Sigma$ with

$$\varphi(x) \in L_\Sigma(x).$$

Appendix 1

Here we recall the concepts of coisotropic submanifolds and symplectic reduction following Weinstein [W].

Appendix 2

We compare different versions of the Maslov index.

2 Basic definitions and generalized fixed point theorems

2.1 Basic definitions and sign conventions

A symplectic manifold (M, ω) is a manifold M together with a closed non degenerate 2–form ω. It is called exact if $\omega = d\alpha$. The main example, in fact, the only one we need, is the cotangent bundle $M = T^*B \stackrel{\pi}{\to} B$ of a given manifold. On cotangent bundles there is a canonical construction to make them symplectic manifolds. In local coordinates (q, p) the canonical one form α is defined as $\alpha = \Sigma p_i dq_i = pdq$ for short. The global definition is as follows: Given $\xi_{q,p} \in T_{q,p}(T^*B)$ we set

$$\alpha(\xi) = p(T\pi(\xi)).$$

We define $\omega = -d\alpha = -d(pdq) = dq \wedge dp$. A diffeomorphism φ is called symplectic if $\varphi^*\omega = \omega$ and exact symplectic if $\varphi^*\alpha - \alpha = dF$ for some function F on M. To a given function (called time dependent Hamiltonian) $H : \mathbb{R} \times M \to \mathbb{R}$ we associate the Hamiltonian vector field X_H defined by

$$i_{X_H}\omega = dH$$

where dH denotes derivative in the space variable. The flow φ_t defined by $\dot{\varphi}_t = X_H(\varphi_t)$ and $\varphi_0 = Id$ is called Hamiltonian isotopy, its time one diffeomorphism we denote by $\varphi_1 = \varphi$. The flow consists of symplectic maps, exact symplectic if M is exact symplectic.

We say that H, φ_t and φ are associated to each other. We say that φ has support in U if $\varphi = Id$ on $M \setminus U$.

We now specify these concepts for the special case $M = \mathbb{R}^{2n}$. With the standard scalar product $< \cdot, \cdot >$ we write $\omega =< J\cdot, \cdot >$ with

$$J := \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}. \tag{1}$$

We write $(q, p) = x$. For $\lambda_x := \frac{1}{2} < Jx, \cdot >$ we have $d\lambda_x = \omega =< J\cdot, \cdot >$. We can write the Hamiltonian equations as

$$\dot{\varphi} = -JH'(\varphi)$$

where H' is the gradient of H taken in space direction.

Example: In $\mathbb{R}^2$ we have for $H(x) = 1/2(q^2 + p^2)$ the flow

$$\varphi_t \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = e^{-Jt} x.$$

□

We define

$$\mathcal{H}^0(\mathbb{R}^{2n}) := \{\varphi \mid \varphi = \varphi_1 \text{ is associated to a compactly supported Hamiltonian function } H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}\}.$$

2.2 Generalized fixed point theorems

We formulate Moser's theorem about generalized fixed points from [M]. We refer to Appendix A for the definition of coisotropic submanifolds.

Theorem 2.1 (Moser) *Let (M, ω) be a simply connected exact symplectic manifold with $\alpha = d\omega$. Let $A \overset{j}{\hookrightarrow} M$ be a compact r–codimensional coisotropic embedded submanifold. Let $\psi : M \to M$ be exact symplectic such that ψ is C^1–close to the identity on a neighbourhood of $j(A)$.*

Then there exist at least two points $x \in M$ such that $\psi(x) \in L_A(x)$, that is x and $\psi(x)$ are on the same leaf in M.

The proof of this theorem shows one of the main reasons why one uses generating functions: One tries to reformulate the intersection problem such that you look for critical points of functions — which are well understood.

Hence we sketch the proof here, but only for $M = \mathbb{R}^{2n}$, thus avoiding all technical difficulties.

Proof:

On $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ we use the symplectic form $\omega \oplus (-\omega)$. Denote by $x = (q, p)$ the coordinates on the second and by $\bar{x} = (\bar{q}, \bar{p})$ coordinates on the first factor.

$$\tilde{\alpha}_{(\bar{x},x)}(\bar{\xi}, \xi) :=< J\bar{x}, \bar{\xi} > - < Jx, \xi >$$

makes $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \omega \oplus (-\omega))$ exact. Define

$$\tilde{\beta}_{(\bar{x},x)}(\bar{\xi},\xi) := < J(\bar{x}-x), \bar{\xi}+\xi > .$$

We have that $\tilde{\beta} = 0$ on the diagonal $\Delta \subset (\mathbb{R}^{2n} \oplus \mathbb{R}^{2n})$ and $\tilde{\beta} - \tilde{\alpha}$ is exact. We define a map $\varphi : A \to A$ such that $\varphi(x) \in L_A(x)$ is the point on $L_A(x)$ closest to $\psi(x)$. (Here we need the smallness condition.)

We look for points with $\psi(x) = \varphi(x)$. Consider the function $\Phi : A \to A \times A$, $x \mapsto (\varphi(x), \psi(x))$. One shows that $\Phi^* \tilde{\beta} = dF$ is exact. The critical points of F are the points we looked for — They exist since A is compact. □

It is a natural task to try to substitute the smallness condition in Moser's theorem by bounds depending on the 'symplectic size' of the submanifold and the symplectic map. To do this we need symplectic invariants, provided by the concept of a capacity. We present the definition by Ekeland and Hofer, [EH] which is applicable for subsets of $\mathbb{R}^{2n}$. We define $B^{2n}(r)$ to be the $2n$–dimensional ball and $Z^{2n}(r)$ the symplectic cylinder

$$Z^{2n}(r) = \{(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 \leq r^2\}$$

Definition 2.2 *A symplectic capacity is a map c which associates to every subset U of $\mathbb{R}^{2n}$ a real number $c(S) \in [0, \infty]$ such that the following axioms hold.*

(A1)	**Normalization:**	$c(B^{2n}(1)) = c(Z^{2n}(1)) = \pi$.
(A2)	**Monotonicity:**	*If* $\varphi \in \mathcal{H}^0(\mathbb{R}^{2n})$ *and* $\varphi(U) \subset V$ *then* $c(U) \leq c(V)$.
(A3)	**Conformality:**	$c(\alpha U) = \alpha^2 c(U)$.

One example of a symplectic capacity is the symplectic width from Gromov, [Gr]. We need the following example of a capacity which has a stronger property than (A2).

Theorem 2.3 (Ekeland/Hofer) *Let Σ be of restricted contact type*[5] *and U be a bounded set such that $\partial U = \Sigma$.*

There exists a symplectic capacity c_{EH} such that $c_{EH}(\Sigma) = c_{EH}(U)$.

[5] see definition A.1

If for a diffeomorphism φ *we furthermore have* $\varphi^*\omega = \alpha\omega$ *it holds that* $c_{EH}(\varphi U) = \alpha^2 c_{EH}(U)$.

In [H] Hofer defines the 'displacement energy' of symplectic diffeomorphisms in $\mathcal{H}^0(\mathbb{R}^{2n})$ to be

$$E_H(\varphi) := \inf\{\sup H - \inf H |\ H \text{ is a Hamiltonian function associated to } \varphi\}.$$

Theorem 2.4 (Hofer) *The map* $d_H : \mathcal{H}^0(\mathbb{R}^{2n}) \times \mathcal{H}^0(\mathbb{R}^{2n}) \to [0,\infty)$ *defined by* $d(\psi,\phi) = E_H(\psi^{-1}\varphi)$ *defines a bi–invariant metric on* $\mathcal{H}^0(\mathbb{R}^{2n})$ *(bi–invariant means for* $\theta \in \mathcal{H}^0(\mathbb{R}^{2n})$ *we have* $d(\theta\psi,\theta\phi) = d(\psi,\phi) = d(\psi\theta,\phi\theta)$*).*

Remark: In the definition of the displacement energy one can use instead of $\sup H - \inf H$ the number $\max_{t\in[0,1]} [\max_x H(t,x) - \min_x H(t,x)]$ or the L^1–norm of the oscillation.

We are now able to formulate Hofer's generalized fixed point theorem.

Theorem 2.5 (Hofer) *Let* Σ *be of restricted contact type and* U *be a bounded set such that* $\partial U = \Sigma$*. Let* $\varphi \in \mathcal{H}^0(\mathbb{R}^{2n})$ *such that* $E_H(\varphi) \leq c_{EH}(\Sigma)$*. Denote by* $L_\Sigma(x)$ *the leaf of the characteristic foliation on* Σ *through* x*, see appendix A.*

Then there exists $x \in \Sigma$ *such that*

$$\varphi(x) \in L_\Sigma(x).$$

Ekeland and Hofer defined their capacity via an infinite dimensional variational principle. In the following section we will define Viterbo's capacities which are constructed with finite dimensional methods.

Hofer asked whether there exists a theorem similiar to theorem 2.5 for Viterbo's capacities instead of Ekeland/Hofer's capacities. In section 6 we will answer this question positively, at least for strictly convex closed hypersurfaces.

3 Generating functions

3.1 Generating functions

If $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symmetric matrix then its graph $\Gamma(P)$ is a Lagrangian subspace in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega) \oplus \omega)$. Hence for a function $S : \mathbb{R}^n \to \mathbb{R}$ the set $\{(x, d_x S) \,|\, x \in \mathbb{R}^n\} \subset T^*\mathbb{R}^n$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$ (Look at the tangent spaces). We call S a naive generating function for $\{(x, d_x S) \,|\, x \in \mathbb{R}^n\}$. By introduction of some auxiliary variables we can generate more general Lagrangians than just 'graphs' in the cotangent bundle. The following definition also works for non trivial vector bundles but since we do not need them we restrict to trivial bundles $B \times \mathbb{R}^N$.

Definition 3.1 *Let $\pi : B \times \mathbb{R}^N \to B$ be a trivial vector bundle on a closed manifold B. Let $S : B \times \mathbb{R}^N := E \to \mathbb{R}$ be a function $(x, v) \to S(x, v) \in \mathbb{R}$ whose fiber derivative is transverse to zero, i.e. for the points of the set*

$$\Sigma_S = \{(x, v) \in E \,|\, \frac{\partial S}{\partial v}(x, v) = 0\}$$

the derivative $T\frac{\partial S}{\partial v}$ has maximal rank. Thus Σ_S is a manifold. Define a map

$$i : \Sigma_S \to T^*B \quad ; \quad (x, v) \mapsto \left(x, \frac{\partial S}{\partial x}\right).$$

S is called a generating function for $L := i(\Sigma_S)$.

Proposition 3.2 *With the above assumptions on the rank $L := i(\Sigma_S) \subset T^*B$ is an immersed Lagrangian manifold.*

<u>Proof:</u>

Consider first the case $M = \mathbb{R}^n$. Then in $T^*(\mathbb{R}^n \times \mathbb{R}^k)$ we consider the vectors that can be written as $(q, v, p, 0)$. They form a coisotropic subspace E_H. The complement ${E_H}^{\omega}$ is given by the vectors of the form $(0, v, 0, 0)$ The reduction $E_H/{E_H}^{\omega}$ hence consists of vectors of the form (q, p) and can be identified with $T^*\mathbb{R}^n$.

The set

$$\tilde{L} := \{(q, v, p, w) \mid p = \frac{\partial S}{\partial q}, \, w = \frac{\partial S}{\partial v}\}$$

is a Lagrangian submanifold of $T^*(\mathbb{R}^n \times \mathbb{R}^k)$.

Since $T\frac{\partial S}{\partial v}$ is non singular we have $T_x\tilde{L} + T_x E_H = T_x T^*(\mathbb{R}^n \times \mathbb{R}^k)$

Hence the reduction of $\tilde{L}$ is Lagrangian and is given by $\pi(\tilde{L} \cap E_H) = i(\Sigma)$.

To see that this works on manifolds as well we only need to observe that E_H can be defined globally as the set of those cotangent vectors that annihilate the kernel of $T\pi$. The reduction of E_H is T^*B. The rest are local constructions. □

We show that we can generate more complex Lagrangians than just graphs:

Example: For $B = \mathbb{R}$ the function $S : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$

$$(x, v_1, v_2) \mapsto -v_1^3/3 + xv_1 - v_2^3/3 + (1 - x)v_2$$

generates an upright figure eight. □

Proposition 3.3 *The circle $S^1 \subset \mathbb{R}^2$ has no generating function*

Sketch of Proof: In section 4.2 we define the Maslov index which associates to every path of Lagrangian sub spaces of $\mathbb{R}^{2n}$ an integer. We now consider an arbitrary Lagrangian submanifold of $\mathbb{R}^{2n}$. If we associate to each closed loop $\gamma(t)$ in $L \subset T^*\mathbb{R}^n$ the path of Lagrangians $T_{\gamma(t)}L$ we get a loop of linear Lagrangians. Associating to this loop its Maslov index we get an integer $\mu(\gamma)$. It turns out that the map $\gamma \to \mathbb{R}$ represents a cohomology class in $H^1(L, \mathbb{Z})$, the Maslov class $\mu(L)$.

It turns out that the Maslov class is invariant under reduction ($\pi^*\mu(\pi L) = \mu(L)$) and is zero for graphs, see C. Viterbo [V1], section 2.

For the path $\gamma(t) = (\sin t, -\cos t)$ we have

$$T_{\gamma(t)}S^1 = \{(r \cos t, r \sin t) \mid r \in \mathbb{R}\}$$

In proposition 4.4 we shall show that its Maslov index is -2. □

3.2 Generating functions quadratic at infinity

The natural question that arises is: When does a given Lagrangian submanifold have a generating function. Up to now there are only partial answers. Since we are interested in critical points of S we only consider generating functions of a special type that have 'sufficiently' many critical points.

If for $||v|| > c$ large $S(x,v) = Q_x(v)$ where Q is a non degenerate quadratic form on each fiber, S is called a *generating function quadratic at infinity* abbreviated by gfqi. We can choose S such that $Q_x = Q$ does not depend on the base variable, see [Th].

The zero section in T^*B has the generating function $S(q) = 0$. We now have the following important existence result proved by C. Sikorav in 1986, [S]:

Theorem 3.4 (Existence, Sikorav) *Let B be a closed manifold. The property of having a gfqi is invariant under Hamiltonian isotopy on T^*B. Thus every Lagrangian submanifold $L \subset T^*B$ which is Hamiltonian isotopic to the zero section has a gfqi.*

See also [Tr] for a proof in the context of paragraph 3.4. □

There are some operations on generating functions leaving L fixed:

- Let $S_1 : E \to \mathbb{R}$, $S_2 : E \to \mathbb{R}$ be two generating functions. If there exists a fiber preserving diffeomorphism
$$\Phi : E \to E; (x,v) \mapsto (x, \varphi(x,v))$$
with $S_2 \circ \Phi = S_1 + \text{const}$ then S_1 and S_2 generate the same Lagrangian immersion and are called equivalent.
- Let $S_1 : E_1 \to \mathbb{R}$ be a generating function and $Q_2 : E_2 \to R$ be a non degenerate quadratic form on the fibers. Then $S_2 = S_1 + Q_2 : E_1 \oplus E_2 \to \mathbb{R}$ generates the same Lagrangian immersion.

If S_2 is obtained from a gfqi S_1 by adding a quadratic form, then S_2 is equivalent to a gfqi, see [Th]. This gfqi is called a *stabilization* of S_1.

We will need that the critical points we are interested in are independent of the gfqi chosen. To do this we need the other important theorem in the theory of generating functions

Theorem 3.5 (Uniqueness, Theret and Viterbo) *Let B be a compact manifold. Let φ be the time one diffeomorphism of a Hamiltonian isotopy on T^*B. Define $L := \varphi(\mathbf{0_B})$ where $\mathbf{0_B} \subset T^*B$ is the zero section. The gfqi for L is unique up to stabilization and equivalence.*

This theorem is due to Viterbo [V2], see also [Th] for a detailed proof. □

3.3 Invariants for Lagrangian submanifolds

Following Viterbo [V2] we may define invariants of Lagrangian submanifolds Hamiltonian isotopic to the zero section and coinciding with the zero section on a open set U (in the next section U will be a neighbourhood of the north pole in S^{2n}). Let $S : E \to \mathbb{R}$ be a gfqi for L, normalized such that $S(x, v) = 0$ for the critical points in U. Define $S^\lambda := \{x \in E | S(x) \leq \lambda\}$.

Since S is quadratic at infinity the homotopy type of the pairs (S^λ, S^μ) and $(S^\mu, S^{-\lambda})$ does not depend on λ for large λ. We may thus write $(S^\infty, S^{-\infty})$ to denote $(S^\lambda, S^{-\lambda})$ for large λ. Denote by D^k the unit ball bundle of dimension k. We have $(S^\infty, S^{-\infty}) \sim (D^{indQ}, S^{indQ-1})$, where $indQ$ is the index of the quadratic form to which S is equal 'at infinity'.

The Thom isomorphism $T : H^*(B) \to H^{*+indQ}(S^\infty, S^{-\infty})$ between cohomologies is shifting the grading by the index of Q[6] . For $u \in H^*(B)$ we define:

$$c(u, L) := \inf\{\lambda \quad | \quad \text{the image of } Tu \text{ under the natural map } H^*(S^\infty, S^{-\infty}) \to H^*(S^\lambda, S^{-\infty}) \text{ is non zero}\}.$$

In Proposition 3.10 we will prove that $c(u, L)$ does not depend on S. Furthermore we have:

Proposition 3.6 *(i) $c(u, L)$ is a critical value for S.*
(ii) Let $(x, v) \in B \times \mathbb{R}^N$ such that $S(x, v) = c(u, L)$. If $d^2S(x, v)$ is non degenerate we have ind $d^2S(x, v) = d +$ ind $Q =: q$ where Q is the quadratic form to which S is equal at infinity and d is the degree of the cohomology class considered.

[6] Since the bundles we are considering are trivial, the Thom isomorphism is given by the Kuenneth formula.

Proof: (i) is an application of the minimax principle. Since S is quadratic at infinity and non degenerate it satisfies the Palais–Smale condition. If $c(u, L) = c$ were not a critical value we would have that $S^{c+\epsilon} \sim S^{c-\epsilon}$, hence $H^*(S^{c+\epsilon}, S^{-\infty}) \cong H^*(S^{c-\epsilon}, S^{-\infty})$. But in $H^*(S^{c+\epsilon}, S^{-\infty})$ the element Tu is non zero and in $H^*(S^{c-\epsilon}, S^{-\infty})$ it is zero.

(ii) We consider a non degenerate critical point on level c. Since we are working in cohomology we can assume that there is only one critical point on level c. Consider the triple $S^{-\infty} \subset S^{c-\epsilon} \subset S^{c+\epsilon}$ and its exact cohomology sequence induced by the natural inclusions

$$\longrightarrow H^q(S^{c+\epsilon}, S^{c-\epsilon}) \xrightarrow{i^*} H^q(S^{c+\epsilon}, S^{-\infty}) \xrightarrow{j^*} H^q(S^{c-\epsilon}, S^{-\infty}) \longrightarrow$$

With the natural maps

$$H^q(S^{\infty}, S^{-\infty}) \xrightarrow{j^*_{\pm}} H^q(S^{c\pm\epsilon}, S^{-\infty})$$

we have $j^* \circ j^*_+(Tu) = j^*_-(Tu)$. By definition it holds that $j^*_-(Tu) = 0$ and $j^*_+(Tu) \neq 0$. Consequently $j^*_+(Tu)$ is in the kernel of j^*. Due to exactness there is a non trivial element in $H^q(S^{c+\epsilon}, S^{c-\epsilon})$.

By looking at S in a Morse chart we see that $(S^{c+\epsilon}, S^{c-\epsilon}) \sim (D^k, S^{k-1})$ where k is the index of the critical point. Hence

$$0 \neq H^q(S^{c+\epsilon}, S^{c-\epsilon}) = H^q(D^k, S^{k-1}) = \begin{cases} \mathbb{R} & q = k \\ 0 & \text{else} \end{cases}$$

Hence $d + \text{ind } Q = q = k$. □

Proposition 3.7 *Denote by $T^*_\Delta(B \times B)$ the restriction of $T^*(B \times B)$ to the diagonal Δ of $(B \times B)$. Let L_1 and L_2 be Lagrangian submanifolds of T^*B.*

Assume that $L_1 \times L_2$ is transverse to $T^*_\Delta(B \times B)$. Define

$$L_1 + L_2 := \{(q, p) \in T^*M \quad | \quad p = p_1 + p_2,\ (q, p_1) \in L_1,\ (q, p_2) \in L_2\}.$$

i) In this situation $L_1 + L_2$ is a Lagrangian submanifold of $T^*(B)$.

ii) Let S_1 and S_2 be generating functions for L_1 and L_2. Define

$$S_1 \sharp S_2(x, v, w) = S_1(x, v) + S_2(x, w).$$

Then $S_1 \sharp S_2$ is a generating function for $L_1 + L_2$.

This is proposition 3.2 in [V2]. □

Proposition 3.8 *Denote by $u \cup v$ the cup product of two cohomology classes u and v. Then*

$$c(u \cup v, S_1 \sharp S_2) \geq c(u, S_1) + c(v, S_2)$$

This is proposition 3.3 in [V2]. the proposition is the crucial tool in computations of the $c(u, L)$. In his diploma thesis, Stefan Born, [Bor] has found a simplified proof which we shall present here. To this aim we first present the following result for the cross product of two cohomology classes.

Proposition 3.9 *Let B_1 and B_2 be two closed manifolds. Let $S_1 + S_2 : E_1 \times E_2 \to \mathbb{R}$ be a generating function for $L_1 \times L_2 \subset T^*(B_1 \times B_2)$. For the cross product $u \times v \in H^*(B_1 \times B_2)$ we have*

$$c(u \times v, S_1 + S_2) = c(u, S_1) + c(v, S_2).$$

Proof of proposition 3.9: Denote by Q_1 and Q_2 the quadratic forms to which S_1 and S_2 are equal at infinity and by q_1 and q_2 the indices of these forms.

We have a commutative diagram

$$\begin{array}{ccc}
H^*(S_1^\lambda \times S_2^\mu, S_1^{-\infty} \times S_2^{-\infty}) & \longleftarrow & H^*((S_1 + S_2)^{\lambda+\mu}, (S_1 + S_2)^{-\infty}) \\
\uparrow & & \uparrow \\
H^*(S_1^\infty \times S_2^\infty, S_1^{-\infty} \times S_2^{-\infty}) & \longleftarrow & H^*((S_1 + S_2)^{\infty}, (S_1 + S_2)^{-\infty}) \\
\uparrow T_1 \times T_2 & & \uparrow T \\
H^{*-q_1-q_2}(B_1 \times B_2) & \stackrel{\cong}{\longleftarrow} & H^{*-q_1-q_2}(B_1 \times B_2)
\end{array}$$

which shows $c(u \times v, S_1 + S_2) \leq c(u, S_1) + c(v, S_2)$.

For the other inequality we assume that $\lambda < c(u, S_1)$ and $\mu < c(v, S_2)$ are not critical values. The long exact sequences of the triples $S_1^{-\infty} \subset S_1^\lambda \subset S_1^\infty$ and $S_2^{-\infty} \subset S_2^\mu \subset S_2^\infty$ guarantee the existence of classes $\tilde{u} \in H^*(S_1^\infty, S_1^\lambda)$ and $\tilde{v} \in H^*(S_2^\infty, S_2^\mu)$ whose images are $T_1 u$ and $T_2 v$.

Their cross product in $H^*(S_1^\infty \times S_2^\infty, S_1^\lambda \times S_2^\infty \cup S_1^\infty \times S_2^\mu)$ is mapped on $T(u \times v)$. Hence we have from the long exact sequence of

$$S_1^{-\infty} \times S_2^{-\infty} \subset S_1^\lambda \times S_1^\infty \cup S_2^\infty \times S_2^\mu \subset S_1^\infty \times S_2^\infty$$

that the image of $T(u\times v)$ in $H^*(S_1^\lambda\times S_2^\infty\cup S_1^\infty\times S_2^\mu, S_1^{-\infty}\times S_2^{-\infty})$ is zero.

The critical points of S_1+S_2 are in a compact set. Hence we can choose K such that

$$H^*((S_1+S_2)^{\lambda+\mu}\cap K,(S_1+S_2)^\infty\cap K)\cong H^*((S_1+S_2)^{\lambda+\mu},(S_1+S_2)^\infty).$$

Considering the inclusions[7]

$$\begin{aligned}&((S_1+S_2)^{\lambda+\mu}\cap K,(S_1+S_2)^{-\infty}\cap K)\\ \overset{j_1}{\hookrightarrow}\ &(S_1^\lambda\times S_2^\infty\cup S_1^\infty\times S_2^\mu, S_1^{-\infty}\times S_2^{-\infty})\end{aligned}$$

and

$$(S_1^\lambda\times S_2^\infty\cup S_1^\infty\times S_2^\mu, S_1^{-\infty}\times S_2^{-\infty})\overset{j_2}{\hookrightarrow}(S_1^\infty\times S_2^\infty, S_1^{-\infty}\times S_2^{-\infty})$$

we get maps

$$\begin{aligned}T(u\times v)\quad &\in\quad H^*(S_1^\infty\times S_2^\infty, S_1^{-\infty}\times S_2^{-\infty})\\ &\xrightarrow{j_2^*}\quad H^*(S_1^\lambda\times S_2^\infty\cup S_1^\infty\times S_2^\mu, S_1^{-\infty}\times S_2^{-\infty})\\ &\xrightarrow{j_1^*}\quad H^*((S_1+S_2)^{\lambda+\mu}\cap K,(S_1+S_2)^{-\infty}\cap K)\\ &\xrightarrow{\cong}\quad H^*((S_1+S_2)^{\lambda+\mu},(S_1+S_2)^{-\infty})\end{aligned}$$

where $j_1^*(T(u\times v))=0$.

Hence $T(u\times v)$ is mapped to zero in $H^*((S_1+S_2)^{\lambda+\mu},(S_1+S_2)^{-\infty})$. Consequently

$$c(u\times v,S_1+S_2)\geq c(u,S_1)+c(v,S_2).$$

□

Proof of proposition 3.8: Consider the commutative diagram

[7] Here our notation hides the problem. S^∞ means S^ν for ν large. $(S_1+S_2)^{\lambda+\mu}$ does not embed in $S_1^\lambda\times S_2^\nu\cup S_1^\nu\times S_2^\mu$ but $(S_1+S_2)^{\lambda+\mu}\cap K$ does.

$$\begin{array}{ccc} (S_1 \sharp S_2)^\lambda & \overset{\Delta}{\hookrightarrow} & (S_1 + S_2)^\lambda \\ \downarrow & & \downarrow \\ (S_1 \sharp S_2)^\infty & \overset{\Delta}{\hookrightarrow} & (S_1 + S_2)^\infty \\ \downarrow \tilde{\pi} & & \downarrow \pi \\ B & \overset{\Delta}{\hookrightarrow} & B \times B \end{array}$$

The Thom isomorphism $T : H^{*-q_1-q_2}(B \times B) \to H^*((S_1 + S_2)^\infty, (S_1 + S_2)^{-\infty})$ is given by $Tu = \pi^* u \cup \theta$ where $\theta \in H^*((S_1+S_2)^\infty, (S_1+S_2)^{-\infty})$ is a Thom class. The restriction of θ to each fibre generates the cohomology of this fibre. Hence $\Delta^*(\theta)$ is a Thom class of $((S_1 \sharp S_2)^\infty, (S_1 \sharp S_2)^{-\infty})$. Note that $u \cup v = \Delta^*(u \times v)$. Consequently

$$\begin{aligned} \Delta^*(\pi^*(u \times v) \cup \theta) &= \Delta^*(\pi^*(u \times v)) \cup \Delta^*(\theta) \\ &= \pi^*(\Delta^*(u \times v)) \cup \Delta^*(\theta) \end{aligned}$$

We get a commutative diagram

$$\begin{array}{ccc} H^*((S_1 \sharp S_2)^\lambda, (S_1 \sharp S_2)^{-\infty}) & \leftarrow & H^*((S_1 + S_2)^\lambda, (S_1 + S_2)^{-\infty}) \\ \uparrow & & \uparrow \\ H^*((S_1 \sharp S_2)^\infty, (S_1 \sharp S_2)^{-\infty}) & \leftarrow & H^*((S_1 + S_2)^\infty, (S_1 + S_2)^{-\infty}) \\ \uparrow T & & \uparrow T \\ H^{*-q_1-q_2}(B) & \leftarrow & H^{*-q_1-q_2}(B \times B). \end{array}$$

If $u \times v$ is mapped to zero on the right hand side it is mapped to zero on the left hand side. Consequently

$$c(u \cup v, S_1 \sharp S_2) \geq c(u \times v, S_1 + S_2) = c(u, S_1) + c(v, S_2).$$

□

Proposition 3.10 *The value $c(u, L)$ of the critical point of S does not depend on the choice of the generating function S.*

Proof: It is clear that $c(u, L)$ is independent of equivalence and adding of a constant. We only have to deal with stabilization. We may assume that S_1 and S_2 are gfqi for L where $S_1 + Q = S_2$ with a quadrati form

Q independent of the base point. We get by proposition 3.9 applied to $B \times pt$:

$$c(u, S_1) = c(u \times 1_{pt}, S_1 + Q) = c(u, S_2)$$

□

Corollary 3.11 *Let* $u, v \in H^*(B)$. *Then*

$$c(u \cup v, \varphi(L)) \geq c(u, L) + c(v, \varphi(\mathbf{0_B})).$$

This is corollary 3.6 in [V2]. □

3.4 Capacities for symplectic diffeomorphisms

Let $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ be a Hamiltonian function with compact support in the $\mathbb{R}^{2n}$–component. The Hamiltonian equations $\dot{\varphi}_t(x) = -JH'(\varphi_t(x))$ define the Hamiltonian flow φ_t. For $t = 1$ we abbreviate the time one diffeomorphism by $\varphi_1 = \varphi$.

If H is time independent then every fixed point $x = \varphi(x)$ corresponds to a periodic orbit of the flow.

We consider Γ_φ, the graph of φ. It is a Lagrangian submanifold of $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega \oplus \omega)$.

We can identify $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega \oplus \omega)$ with $(T^*\mathbb{R}^{2n}, \omega)$ through the map

$$\tau : (q, p, \bar{q}, \bar{p}) \mapsto \left(\frac{q + \bar{q}}{2}, \frac{p + \bar{p}}{2}, \bar{p} - p, q - \bar{q} \right). \tag{2}$$

Note that for the diagonal $\Delta \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ we have $\tau(\Delta) = \mathbf{0}_{T^*\mathbb{R}^{2n}}$. Hence $\tau(\Gamma_\varphi)$ coincides with the zero section outside a compact set since H is compactly supported. Considering $\mathbb{R}^{2n}$ as the punctured sphere $S^{2n} \setminus \{P\}$, P the north pole, we get symplectic embedding $T^*\mathbb{R}^{2n} \to T^*S^{2n}$: Any diffeomorphism $\mathbb{R}^{2n} \to S^{2n} \setminus \{P\}$ can be considered as a chart of S^{2n}. The associated chart of the cotangent bundle constitutes the desired symplectic embedding $T^*\mathbb{R}^{2n} \to T^*S^{2n}$. We can compactify the image of $\tau(\Gamma_\varphi)$ by adding the point $(P, 0)$.

We thus get a Lagrangian manifold $\tilde{\Gamma}_\varphi$ in T^*S^{2n}. This manifold is isotopic to the zero-section via $\tilde{\varphi} := \tau(id \times \varphi)\tau^{-1}$ extended to T^*S^{2n}. This $\tilde{\varphi}$ is the time one diffeomorphism of the Hamiltonian $(0 \oplus H)\tau^{-1}$.

By the existence and uniqueness theorems $\tilde{\Gamma}_\varphi$ has a unique generating function quadratic at infinity normalized such that the critical value corresponding to the pole is zero. To every critical point (x, v) of this gfqi there corresponds a fixed point x of φ: $x = \pi_1 \tau(x, x)$ where $\pi_1 : T^*S^{2n} \to S^{2n}$ is the projection map.

Definition 3.12 *For* $1 \in H^0(S^{2n})$ *and* $\mu \in H^{2n}(S^{2n})$*,the orientation class, we define the* capacities *of symplectic diffeomorphisms to be*

$$c_+(\varphi) = -c(1, \tilde{\Gamma}_\varphi); \quad c_-(\varphi) = -c(\mu, \tilde{\Gamma}_\varphi); \quad \gamma(\varphi) = c_+(\varphi) - c_-(\varphi).$$

The reason for the minus sign is to get a suggestive formulation for a positive Hamiltonian, see the next Proposition, part (vii): If $\varphi \neq id$ is associated to a Hamiltonian $H \geq 0$, then $c_+(\varphi) > 0$, $c_-(\varphi) = 0$.

There is a set of propositions we shall need to compute the capacities in the proofs of theorem 5.4 and 6.1 later on.

Proposition 3.13 *Let* φ *and* ψ *be symplectic diffeomorphisms with compact support. The symplectic invariants* c_+*,* c_- *and* γ *have the following properties:*

(i) $c_+(\varphi) \geq 0$ *and* $c_-(\varphi) \leq 0$ $\quad \gamma(\varphi) = 0 \iff \varphi = id$
(ii) $c_+(\varphi) = -c_-(\varphi^{-1})$
(iii) $c_+(\psi\varphi) \leq c_+(\psi) + c_+(\varphi)$
(iv) $c_-(\psi\varphi) \geq c_-(\psi) + c_-(\varphi)$
(v) $c_-(\psi\varphi) \leq c_-(\psi) + c_+(\varphi)$
(vi) $c_+(\psi\varphi) \geq c_+(\psi) + c_-(\varphi)$
(vii) Let ψ_t *be a conformal symplectic isotopy, i.e. we assume* $\psi_t^*\omega = \lambda(t)\omega$ *for a differentiable function* $\lambda(t) > 0$*. Assume that* $\psi_0 = id$*. We then have* $c_\pm(\psi_t \varphi \psi_t^{-1}) = \lambda^2(t) c_\pm(\varphi)$*.*

In particular we conclude that $d(\psi, \varphi) := \gamma(\psi^{-1}\varphi)$ defines a bi–invariant metric on $\mathcal{H}^0(\mathbb{R}^{2n})$.

Proof: These relations are proved in [V2] except for (v) and (vi), which can nevertheless be proved similarly to (iii): We have $\tilde{\Gamma}_{\psi\varphi} = \tilde{\psi}\tilde{\Gamma}_\varphi$ and therefore

$$\tilde{\psi}\tilde{\Gamma}_\varphi = \tilde{\psi}\tau\Gamma_\varphi = (\tau \circ (id \times \psi)\tau^{-1})\tau\Gamma_\varphi = \tau\Gamma_{\psi\varphi} = \tilde{\Gamma}_{\psi\varphi}.$$

We thus have in view of corollary 3.11

$$\begin{aligned} -c_-(\psi\varphi) &= c(1\cup u, \tilde{\Gamma}_{\psi\varphi}) \\ = c(1\cup u, \tilde{\psi}\tilde{\Gamma}_\varphi) \geq c(1,\tilde{\Gamma}_\varphi) + c(u, \tilde{\psi}(\mathbf{0}_B)) &= c(1,\tilde{\Gamma}_\varphi) + c(u,\tilde{\Gamma}_\psi) \\ &= -c_+(\varphi) - c_-(\psi). \end{aligned}$$

Finally, (vi) follows from (v) and (ii). □

Proposition 3.14 *The following relations hold:*
(i) Let $H_1 \leq H_2$. For the associated time one diffeomorphisms φ^1, φ^2 we have

$$c_\pm(\varphi^1) \leq c_\pm(\varphi^2).$$

(ii)Let ψ and φ be symplectic diffeomorphisms with compact support such that the support of φ is in U and $\varphi(U)\cap U = \emptyset$. We then have

$$c_\pm(\varphi\psi) = c_\pm(\varphi) \text{ and } c_+(\psi) \leq \gamma(\varphi).$$

See corollary 4.5 and proposition 4.6 in [V2]. □

We have two kinds of continuity.

Proposition 3.15 *(i) Let H_1 and H_2 be two compactly supported Hamiltonians. Let φ^1 and φ^2 be the associated time one diffeomorphisms. If $|H_1 - H_2|_{C_0} \leq \epsilon$ then we have $|\gamma(\varphi^1) - \gamma(\varphi^2)| \leq \epsilon$. The same holds for c_+ and c_-.*
(ii) c_+, c_- and γ are continuous on $\mathcal{H}^0(\mathbb{R}^{2n})$ for the C^0–topology for symplectic diffeomorphisms.

See proposition 4.14 and 4.15 in [V2]. □

Assume that φ is the time one diffeomorphism of a compactly supported Hamiltonian H. Let z be a point outside the support of H. Then $S(z,0) = 0$ where S is a gfqi for φ. There exist critical points $(x_\pm, v) \in \mathbb{R}^{2n}\times\mathbb{R}^N \subset S^{2n}\times\mathbb{R}^N$ of S such that

$$c_\pm(\varphi) = -S(x_\pm, v) = -S(x_\pm, v) + S(z,0)$$

We have the following representation formula which shows that the capacities are classical actions of certain paths.

Proposition 3.16 *If $g_\pm : [0,1] \to S^{2n}$ is a smooth path connecting z with x_+ (respectively x_-) then*

$$c_\pm(\varphi) = \int_{g(t)} pdq - \int_{\varphi(g(t))} pdq = -\left(\int_{\varphi_t(x_\pm)} pdq - H\,dt \right).$$

Proof: This is proposition 4.2(3) in [V2]. Since the proof of the first equality is skipped in Viterbo's papers we perform it here. To simplify notation we only proof it for c_+ with $x = x_+$. Let S be a gfqi for φ. Denote by $\hat{g}$ a path from $(z,0)$ to (x,v) such that $\hat{g}(t) \in \Sigma_S$[8] and $i(\hat{g}(t)) = g(t)$.

We have

$$\begin{aligned} c_+(\varphi) &= -S(x,v) + S(z,0) = -S(\hat{g}(1)) + S(\hat{g}(0)) \\ &= -\int_0^1 < dS(\hat{g}(t)) | \dot{\hat{g}}(t) > \, dt. \end{aligned}$$

Consider the path $\hat{\gamma}(t) = (\hat{g}(t), dS(\hat{g}(t))) \in T^*(S^{2n} \times \mathbb{R}^N)$. Its tangential is given by

$$T\hat{\gamma}(t) = (\hat{g}(t), dS(\hat{g}(t)), \dot{\hat{g}}(t), d^2S(\hat{g}(t)) \cdot \dot{\hat{g}}(t)).$$

With the Liouville form $\hat{\lambda} = \hat{p}d\hat{q}$ on $T^*(S^{2n} \times \mathbb{R}^N)$ we have

$$c_+(\varphi) = -\int_{\hat{\gamma}} \hat{\lambda} = -\int_{\hat{\gamma}-\hat{\gamma}_0} \hat{\lambda} = \int_{\hat{D}} \hat{\omega}$$

where $\hat{\gamma}_0 = (\hat{g}(t), 0)$, $\hat{\omega}$ the symplectic form on $T^*(S^{2n} \times \mathbb{R}^N)$ and $\hat{D}$ a disc bounded by $\hat{\gamma} - \hat{\gamma}_0$ with the right orientation.

Considering the symplectic reduction $\pi : E_H \to T^*\mathbb{R}^{2n}$ we observe that $\tilde{\Gamma}_\varphi = \tau\Gamma_\varphi$ is the image of the manifold

$$\{(x,v), dS(x,v)\}_{(x,v)\in T^*(\mathbb{R}^{2n}\times\mathbb{R}^N)} \cap E_H$$

and $\mathbf{0}_{T^*\mathbb{R}^{2n}}$ is the image of $\mathbf{0}_{T^*(\mathbb{R}^{2n}\times\mathbb{R}^N)}$.

For elements α_1 and α_2 in $T^*(\mathbb{R}^{2n} \times \mathbb{R}^N)$ we have $\hat{\omega}(\alpha_1, \alpha_2) = \tilde{\omega}(\pi\alpha_1, \pi\alpha_2)$ where $\tilde{\omega}$ is the symplectic form on $T^*\mathbb{R}^{2n}$. Applying τ^{-1} yields the following formula on $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega \oplus \omega)$.

$$c_+(\varphi) = \int_D (-\omega \oplus \omega)$$

[8] Σ_S is the critical set along the fibers, see definition 3.1.

In this formula D is a disc bounded by $\gamma - \gamma_0$, the projection of $\hat{\gamma} - \hat{\gamma}_0$. We have that $\gamma(t) = (g(t), \varphi(g(t))$ and $\gamma_0(t) = (g(t), g(t))$.

Indicating by $\bar{q}$ and $\bar{p}$ coordinates on the second $\mathbb{R}^{2n}$ we conclude

$$c_+(\varphi) = \int_D (-\omega \oplus \omega) = -\int_{\gamma-\gamma_0} -pdq + \bar{p}d\bar{q}.$$

Since γ_0 does not give a contribution we have

$$c_+(\varphi) = \int_{g(t)} pdq - \int_{\varphi(g(t))} pdq.$$

For the second equality of the proposition we are reduced to computations on $\mathbb{R}^{2n}$.

To apply Stokes' theorem once more we consider $\Phi : (s,t) \mapsto \varphi_t(g(s))$. The four paths

$$\begin{array}{lll} \gamma_1 : & s \mapsto & g(s) = \Phi(s,0) \\ \gamma_2 : & t \mapsto & \varphi_t(x) = \Phi(1,t) \\ \gamma_3 : & s \mapsto & \varphi_1 g(s) = \Phi(s,1) \\ \gamma_4 : & t \mapsto & \varphi_t(z) = \Phi(0,t) = z \end{array}$$

bound Φ i.e. $\gamma_1 + \gamma_2 - \gamma_3 - \gamma_4 = \partial(\Phi([0,1]^2))$.

We have $c_+(\varphi) = +\int_{\gamma_1} pdq - \int_{\gamma_3} pdq$ and $\int_{\gamma_4} pdq = 0$.

Furthermore we have

$$\int_{\partial([0,1]^2)} (\Phi(s,t),t)^* H(\Phi(s,t),t)dt = \int_{(\gamma_2(t))} Hdt$$

since the first and third path do not depend on t and the fourth path is constant.

We conclude

$$\begin{aligned} c_+(\varphi) &= \int_{\partial([0,1]^2)} (\Phi(s,t),t)^*(pdq - Hdt) - \int_0^1 (\Phi(1,t),t)^*(pdq - Hdt) \\ &= \int_{([0,1]^2)} (\Phi(s,t),t)^* d(pdq - Hdt) - \int_0^1 (\varphi_t(x),t)^*(pdq - Hdt). \end{aligned}$$

It is left to show that $\int_{([0,1]^2)} (\Phi(s,t),t)^* d(pdq - Hdt) = 0$. Define $\Psi = (\Phi, t)$. We have

$$(\Phi(s,t),t)^* d(pdq - Hdt) = i_{\frac{\partial \Psi}{\partial s}} i_{\frac{\partial \Psi}{\partial t}} (d(pdq - Hdt))(dt \wedge ds).$$

With $\frac{\partial \Psi}{\partial t} = (\dot{\varphi}_t, 1) = (X_H(\varphi(t), 1)$ we compute

$$\begin{aligned} i_{(\frac{\partial \Psi}{\partial t})}(dp \wedge dq - dH \wedge dt) &= -\omega(X_H, \cdot) - dH(X_H)dt + dH \cdot 1 \\ &= -\omega(X_H, \cdot) - \omega(X_H, X_H)dt + \omega(X_H, \cdot) \\ &= 0. \end{aligned}$$

□

Furthermore we shall need later on the following proposition from [Bor]:

Proposition 3.17 (Born) *Let $H(x,t)$ be a Hamiltonian function on $[0,1] \times \mathbb{R}^{2n}$ with associated flow φ_t. We then have*

$$\begin{aligned} c_+(\varphi) &\leq \sup_{x,t} H(x,t) \\ c_-(\varphi) &\geq \inf_{x,t} H(x,t) \\ \gamma(\varphi) &\leq \sup_{x,t} H(x,t) - \inf_{x,t} H(x,t). \end{aligned}$$

Proof: We prove the statement for c_+. For every $k \in \mathbb{N}$ we can break the flow up to get

$$\varphi_1 = (\varphi_1 \circ \varphi^{-1}_{1-1/k}) \circ (\varphi_{1-1/k} \circ \varphi^{-1}_{1-2/k}) \ldots (\varphi_{1/k} \circ I).$$

By proposition 3.13 (iii) we have

$$c_+(\varphi) \leq \sum_{i=1}^{k} c_+(\varphi_{i/k} \circ \varphi^{-1}_{i-1/k}).$$

One observes that $\psi^i := \varphi_{i/k} \circ \varphi^{-1}_{i-1/k}$ is the time one diffeomorphism of

$$H_i(x,t) := \frac{1}{k} H\left(x, \frac{(i-1)+t}{k}\right).$$

We find fixed points x_i of ψ^i such that for $x_i(t) := \psi^i_t(x)$ we have

$$\begin{aligned} c_+(\psi^i) &= \int_0^1 \frac{1}{2} < Jx_i, \dot{x}_i > dt + \int_0^1 H_i(x_i, t) dt \\ &\leq \frac{1}{2} \int_0^1 ||x_i|| \cdot ||\dot{x}_i|| dt + \sup H_i \\ &\leq \frac{1}{2} \sup(H_i') \int_0^1 ||x_i|| dt + \sup H_i \\ &\leq \frac{1}{2} \sup(H_i') \cdot \sup(H_i') + \sup H_i \\ &\leq \frac{1}{2k^2} \sup ||H'||^2 + \frac{1}{k} \sup H \end{aligned}$$

Consequently,

$$c_+(\varphi) \leq \frac{1}{2k} \sup ||H'||^2 + \sup H$$

and taking the limit $k \to \infty$ the first claim follows. The second statement is proved in the same way and the third statement follows from the definition. □

From proposition 3.17 we conclude immediately that $\gamma(\varphi) \leq E(\varphi)$ where E is Hofer's diplacement energy from theorem 2.4.

Definition 3.18 *For a compactly supported Hamiltonian* $H : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ *with associated flow* φ_t *and fixed point* x *of* φ_1 *the Hamiltonian action is*

$$\begin{aligned} \mathcal{A}_H(x) &= \int_{\varphi_t(x)} pdq - H\,dt = \int_{\varphi_t(x)} -\frac{1}{2}\lambda - H\,dt \\ &= \int_{\varphi_t(x)} \frac{1}{2} < -Jx, \cdot > -H\,dt. \end{aligned}$$

The action spectrum is:

$$\mathcal{A}(H) := \left\{ \int_{\varphi_t(x)} \lambda - H\,dt \,|\, \varphi_t(x) \textit{ is a fixed point of } \varphi \right\}.$$

Proposition 3.16 shows that the capacities are represented by the Hamiltonian action of a Hamiltonian function. The Hamiltonian action is an important, well–studied, subject in symplectic geometry. We have the following propositions:

Proposition 3.19 *If H and K generate the same time one diffeomorphism φ we have for a fixed point x*

$$\mathcal{A}_H(x) = \mathcal{A}_K(x).$$

This is Lemma 5.1 in [HZ]. □

Remark : For φ^1, φ^2 with associated Hamiltonians H_1 and H_2 we consider $\varphi = \varphi^2 \circ \varphi^1$. We can view φ as time two diffeomorphism for the Hamiltonian H with $H(t,x) = H_1(t,x)$ for $t < 1$ and $H(t,x) = H_2(t-1,x)$ for $t > 1$. In our context it does not matter that H is not continuous for $t = 1$. We have for

$$x(t) = \begin{cases} \varphi_t^1(x) \text{ for } t \in [0,1] \\ \varphi_{t-1}^2(\varphi_1^1)(x) \text{ for } t \in [1,2] \end{cases}$$

$$\begin{aligned} c_\pm(\varphi) &= -\left(\int_{\varphi_t^1(x_\pm)} \lambda - H_1\, dt + \int_{\varphi_t^2(\varphi_1^1(x_\pm))} \lambda - H_2\, dt \right) \\ &= -\left(\int_0^2 x^*\lambda - \int_0^1 H_1(x(t),t)\, dt - \int_1^2 H_2(x(t),t)\, dt \right). \end{aligned}$$

Proposition 3.20 *The action spectrum is compact and nowhere dense.*

This is proposition 5.8 in [HZ]. □

The previous proposition is a massive restriction of the values the capacities $c_\pm(\varphi)$ can take. Together with the continuity of the capacity (proposition 3.15) the above proposition is a useful tool to compute capacities as we will see in the proofs of theorem 5.4 and 6.1.

On account of Proposition 3.14 (ii) there is an upper bound for $c_+(\varphi)$ with support in U given by $\gamma(\phi)$, therefore the following capacity is $c(U)$ is finite for bounded open sets U.

Definition 3.21 *For open sets U we define the* capacity

$$c(U) := sup\{c_+(\varphi) \,|\, \varphi \text{ has compact support in } U\}$$

This can be used to define the capacity for arbitrary sets

Definition 3.22 *The capacity of an arbitrary set $\Sigma \subset \mathbb{R}^{2n}$ is*

$$c(\Sigma) := \inf\{c(U) | U \text{ open and bounded}, \Sigma \subset U\}$$

Proposition 3.23 *$c : \Sigma \to c(\Sigma) \in [0, \infty]$ defines a capacity in the sense of the axioms of definition 2.2.*

Proof: The monotonicity axiom is clearly satisfied. In the next section we show that the normalization axiom is satisfied. As for the conformality we observe that to every $\tilde{\varphi} \in \mathcal{H}^0(\mathbb{R}^{2n})$ with support in U we find an associated Hamiltonian $\tilde{H}$ with support in U and a time independent Hamiltonian $H \geq \tilde{H}$. We can deform H to a Hamiltonian with support in αU via the homotopy

$$H^s(x) = (1 - (1 - \alpha)s)^2 H((1 - (1 - \alpha)s)^{-1} x).$$

and prove

Lemma 3.24 *Denote by φ^α the time one diffeomorphism of H^1. It holds that*

$$c_+(\varphi^\alpha) = \alpha^2 c(\varphi).$$

Proof of lemma 3.24: If $x(t)$ is a periodic orbit of φ then $x_\alpha(t) = \alpha x(t)$ is a periodic orbit of φ^α:

$$\alpha \dot{x}(t) = \alpha \dot{\varphi}_t(x) = -\alpha J H'(\varphi_t(x))$$

$$= -\alpha J H'(\alpha^{-1} \alpha \varphi_t(x)) = -J H'_\alpha(\alpha \varphi_t(x))$$

Considering the Hamiltonian action we get

$$\begin{aligned}
\Phi_{H_\alpha}(\alpha x) &= \int (-\omega(\alpha x(t), \alpha \dot{x}(t)) - H_\alpha(\alpha x(t)))\, dt \\
&= \alpha^{-2} \int (-\omega(x(t), \dot{x}(t)) - H(x(t)))\, dt \\
&= \alpha^{-2} \Phi_H(x)
\end{aligned}$$

Since the action spectrum is compact and nowhere dense and since the capacity depends continuously on the Hamiltonian we have $c_+(\varphi^\alpha) = \alpha^{-2}c_+(\varphi)$. □

The proof of proposition 3.23 is now easy: Since $\varphi \leftrightarrow \varphi^\alpha$ defines a one to one correspondence between symplectic diffeomorphisms having support in U and in αU, we have $c(\alpha U) = \alpha^2 c(U)$. The proof of proposition 3.23 is finished. □

In theorem 5.4 we show that for strictly convex hypersurfaces Σ and bounded sets U with $\partial U = \Sigma$ we have $c(U) = c(\Sigma)$. In the next section we give an explicit construction for the special case $\Sigma = S^{2n-1}$ and show that $c(S^{2n-1}) = c(B(1)) = \pi$. Thus Viterbo's capacities are non trivial and normalized.

3.5 Viterbo's capacity of the sphere

This section gives an explicit construction to prove that the Viterbo–capacity of the sphere is π.

For the disk of radius r we have:

Proposition 3.25 (Viterbo) $c(B^{2n}(r)) = \pi r^2$.

We prove this proposition together with the following proposition:

Proposition 3.26 $c(S^{2n-1}(r)) = \pi r^2$.

Furthermore we show that the capacity of the symplectic cylinder

$$Z^{2n}(r) = \{(x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 \le r^2\}$$

is πr^2 as well.

Proof: We only consider $r = 1$. Denote by $B(1,1)$ the disc of radius 1, centered at $(1,1)$.

We first construct a Hamiltonian h such that for its time one diffeomorphism φ we have

$$\varphi(B(1,1)) \cap (B(1,1)) = \emptyset$$

and $\gamma(\varphi) \leq \pi + \epsilon$ so that by proposition 3.14 the capacity of the disc is estimated by $c(B(1,1)) \leq \pi r^2$.

Consider $f(y) = 2\sqrt{1+\epsilon-(y-1)^2}$. For a function F with $F' = f$ and $F(0) = \epsilon$ we have $F(2) = \pi + \tilde{\epsilon}$. Define

$$h(x,y) = \begin{cases} F(y) \text{ on } [0,2]^2 \\ 0 \text{ on } \mathbb{R}^2 \setminus [-\epsilon, 2+\epsilon]^2 \\ \text{smooth in between} \end{cases}$$

such that $\frac{\partial h}{\partial y}(x, 2+\epsilon/2) = 0$ for $x \in [0,2]$ and $h(x,y) \geq 0$. We have $h(x, 2+\epsilon/2) = \pi + \tilde{\epsilon}$.

The only 1-periodic orbits of φ_t are the equilibrium points $(x, 2+\epsilon/2)$ where the derivative of h vanishes. By definition of the capacities $c_+(\varphi) = \pi + \tilde{\epsilon}$ and $c_-(\varphi) = 0$. Since ϵ and $\tilde{\epsilon}$ are arbitrary small we conclude $c(B(1,1)) \leq \pi$.

We next show that for the symplectic cylinder Z^{2n} of radius 1 we have $c(Z^{2n}) \leq \pi$.

For a bounded subset $U \subset Z^{2n}$ of diameter at most d consider the time one diffeomorphism of

$$H(x_1, \dots, x_n, y_1, \dots, y_n) = g(|(x_2, \dots, x_n, y_2, \dots, y_n)|) \cdot h(x_1, y_1)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth function with slope smaller than $2\pi \cdot (2R)$such that

$$g = \begin{cases} 1 \text{ on } [0, 2R] \\ 0 \text{ on } [3R, \infty[\end{cases}$$

where $R > \pi$ and $R > d$.

The time one diffeomorphism ψ of H satisfies $\psi(U) \cap U = \emptyset$, and the only one periodic orbits are as above equilibrium points having the action $\pi + \epsilon$.

We conclude by definition of the capacity that $c(Z^{2n}) \leq \pi$.

We next prove that $c(B^{2n}(1)) \geq \pi$.

Given $\delta \in]0,1[$ we define a function $g : (0,\infty) \to \mathbb{R}$:

$$g_\delta(s) := \begin{cases} 0 & \text{for } s > 1+\delta, s < 1-\delta \\ 2\pi(1-\delta) & \text{for } s = 1 \\ \text{affine linear in between} & \end{cases}$$

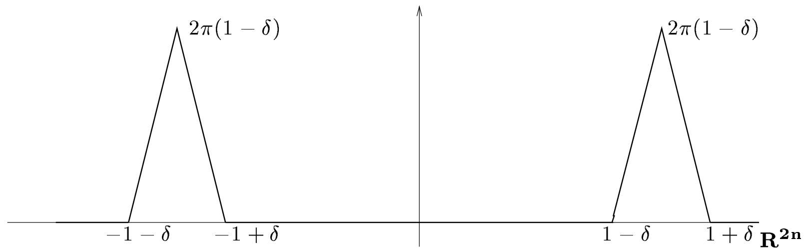

Figure 1: H_δ

Define $H_\delta(x) = g_\delta(|x|)$ (see figure 1).

In the linear part of H_δ the slope is $\pm\frac{2\pi(1-\delta)}{\delta}$. Since H_δ is not smooth we can consider the Hamiltonian equations in the linear parts only

The length of a periodic orbit of the Hamiltonian flow is $2\pi n|x|$. We are interested in one periodic orbits. Hence the length of the orbit has to be equal to the slope. The condition we get is thus:

$$\frac{2\pi(1-\delta)}{\delta} = 2n\pi(1 \pm b), \quad b \in]0, \delta[,$$

i.e.

$$1/n \in \left] \frac{(1-\delta)\delta}{1-\delta} = \delta\,,\ \frac{(1+\delta)\delta}{1-\delta} \right[\setminus \left\{ \frac{\delta}{1-\delta} \right\} =: I.$$

We find a sequence δ_n such that there are no non constant 1–periodic orbits for $\varphi_{H_{\delta_n}}$:

For every $n \in \mathbf{N}$ we take $\delta_n = \frac{1}{n+1}$, so $\frac{1}{n+1} \notin I$ and

$$n = \frac{1}{\delta} - 1 \iff \frac{1}{n} = \frac{\delta}{1-\delta} \notin I.$$

Furthermore we have to prove: $\frac{1}{n-1} \geq \frac{(1+\delta_n)\delta_n}{1-\delta_n} \iff 1 - 2\delta_n \leq \frac{1-\delta_n}{(1+\delta_n)} \iff 1 - \delta_n - 2\delta_n^2 \leq 1 - \delta_n$ which is true, so that we have found the desired sequence.

For every δ in our sequence we will construct a smooth version of H_δ (which we will call H_δ, too) that has no periodic orbits $x(t)$ with

$$-\mathcal{A}_{H_\delta}(x) := \int_0^1 1/2 < Jx, \dot{x} > dt + \int_0^1 H_\delta(x(t))dt \in]0, \pi - \epsilon[$$

where $\epsilon \to 0$ as $\delta \to 0$. Since the flow φ_{H_δ} of H_δ is non trivial, $c_+(\varphi_{H_\delta}) > 0$. Since $c(S^{2n-1}) \leq c(B^{2n}) = \pi$ this will prove $c(S^{2n-1}) = \pi$.

The smoothened version of g_δ differs from the original one in the following four regions (see figure 2). Here a and b are real numbers with $a << \delta^2/2$, $b = 2\pi a \frac{1-\delta}{\delta}$.

1. $]1-\delta, 1-\delta+a[$ with $g'_\delta > 0$, $g_\delta(1-\delta+a) = b$
2. $]1-a, 1[$ with $g'_\delta > 0$, $g_\delta(1-a) = 2(1-\delta)\pi - b$
3. $]1, 1+a[$ with $g'_\delta < 0$, $g_\delta(1+a) = 2(1-\delta)\pi - b$
4. $]1+\delta-a, 1+\delta[$ with $g'_\delta < 0$, $g_\delta(1+\delta-a) = b$

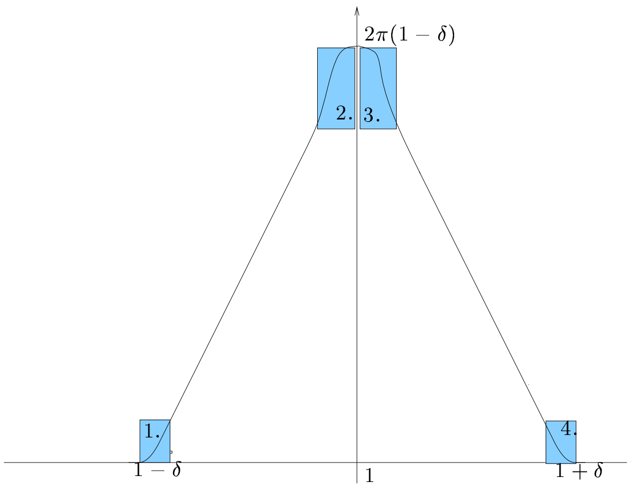

Figure 2: g_δ, smoothened

For a 1–periodic solution we must have

$$|H'_\delta(x)| = |g'_\delta(|x|)x/|x|\,| = |g'_\delta(|x|)| = n2\pi|x|$$

for some $n > 0$ (the solution is not constant).

Region 1: Here we have

$$\begin{aligned} -\mathcal{A}_{H_\delta}(x) &= -(\int 1/2 < -Jx, \dot{x} > - \int H_\delta(x(t))dt) \\ &= \int 1/2 < Jx, -Jg'_\delta(|x|)x/|x| > + \int H_\delta(x(t))dt \\ &= \int -1/2 < x, g'_\delta(|x|)x/|x| > + \int H_\delta(x(t))dt \\ &= -g'(|x|)|x| + d \\ &= -n\pi|x|^2 + d < 0 \end{aligned}$$

with $d \in [0, b]$.

Region 2: $-\mathcal{A}_{H_\delta}(x) = -n\pi|x|^2 + 2\pi(1-\delta) - d$ with $|x| = 1-e$, $e \in [0, a]$. For $n = 1$ we have $-\mathcal{A}_{H_\delta} = -\pi(1-e)^2 + 2\pi(1-\delta) - d > \pi - \epsilon$. For $n \geq 3$ we have $-\mathcal{A}_{H_\delta} < 0$.

For $n = 2$:

$$\begin{aligned} -\mathcal{A}_{H_\delta}(x) &= -2\pi(1-e)^2 + 2\pi(1-\delta) - d \\ &= 2\pi(2e - e^2 - \delta) - d < 0 \end{aligned}$$

Region 3: Here $g'_\delta < 0$. Hence $g'_\delta = -2n\pi|x|$.

$-\mathcal{A}_{H_\delta}(x) = n\pi|x|^2 + 2\pi(1-\delta) - d > 3\pi - \pi\delta - d > 2\pi$.

Region 4: $-\mathcal{A}_{H_\delta}(x) = n\pi|x|^2 + d > \pi + \epsilon$.

Hence φ_{H_δ} has no fixed point with Hamiltonian action in the interval $]0, \pi - \epsilon[$. Consequently $c_+(\varphi_{H_\delta}) \geq \pi - \epsilon$. Taking the limit $\delta \to 0$ (and $\epsilon \to 0$) we get the desired estimate. □

4 The Maslov index

In the last section we associated to a symplectic diffeomorphism φ the Lagrangian submanifold $\tau\Gamma(\varphi)$ and found a generating function S for $\tau\Gamma(\varphi)$. To fixed points x and y of φ there correspond critical points (x, v) and (y, w) of S.

In [V1] C. Viterbo observed that the difference

$$\text{ind } d^2S(x, v) - \text{ind } d^2S(y, w)$$

is independent of the particular generating function chosen and related to the Conley–Zehnder index defined in [CZ].

Theret in [Th] used this index in the more general setting of paths of Lagrangians. He proved directly that this index satisfies the axioms of a Maslov index as defined by Cappell, Lee and Miller in [CLM].

In fact there are several possibilities to generalize the Maslov index as defined in [A]. In appendix B we compare some indices.

In this section we describe Theret's 'generating function approach' to the index and prove a 'generic formula'.

4.1 Linear Lagrangians

Let $L \subset T^*\mathbb{R}^n$ be a Lagrangian and $S : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ a generating function for L. Remember that Σ_S is the critical locus of fibre critical points of S and $i_S : \Sigma_S \to T^*\mathbb{R}^n$ is a Lagrangian immersion such that $i_S(\Sigma_S) = L$.

We need two preliminary observations.

Proposition 4.1 *Let $S : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$ be a generating function for $L \subset \mathbb{R}^{2n}$. If $(x, v) \in \Sigma_S$ with $i_S(x, v) = z \in L$ then for $(r, s) \in \mathbb{R}^n \times \mathbb{R}^N$*

$$Q : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R} \qquad ; \qquad (r, s) \mapsto d^2S(x, v).(r, s).(r, s)$$

is a generating function for T_zL.

Proof: T_zL is the image of

$$di_S(x, v) : T_{(x,v)}\Sigma \to T_z\mathbb{R}^{2n}.$$

We have to describe $T_{(x,v)}\Sigma$ and $di_S(x,v)$. Observe that $dQ(r,s).(\tilde{r},\tilde{s}) = 2d^2S(x,v).(r,s).(\tilde{r},\tilde{s})$

We have defined Σ_S to be

$$\begin{aligned}\Sigma_S &= \{(x,v) \mid \frac{\partial S}{\partial v}(x,v) = 0\} \\ &= \{(x,v) \mid dS(x,v).(0,\tilde{s}) = 0, \ \forall \tilde{s} \in \mathbb{R}^N\}.\end{aligned}$$

Hence

$$T_{(x,v)}\Sigma = \{(r,s) \mid d^2S(x,v).(r,s).(0,\tilde{s}) \ \forall \tilde{s} \in \mathbb{R}^N\} = \{(r,s) \mid \frac{\partial Q}{\partial s} = 0\}.$$

Furthermore $i_S(x,v) = (x, \frac{\partial S}{\partial x}(x,v)) = dS(x,v).(.,0)$. Hence

$$di_S(x,v)(r,s) = (r, d^2S(x,v).(r,s).(.,0)) = (r, \frac{\partial Q}{\partial r}).$$

□

Denote by $\Lambda(n)$ the set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega)$ with the standard symplectic form $\omega =< J\cdot, \cdot >$. The Maslov index will associate to every path $\gamma : [a,b] \to \Lambda(n)$ an integer.

To this end we need quadratic generating functions (generating forms) for $\gamma(t)$. The problem is that our existence and uniqueness theorems are proved so far for paths of Lagrangian submanifolds coinciding with the zero section outside a compact set and starting from the zero section.

Since $\Lambda(n)$ is path connected we can introduce a new path $\tilde{\gamma} : [c,b] \to \Lambda(n)$, $c < a$ with $\tilde{\gamma}(c) = \mathbb{R}^n \times \{0\}$ and $\tilde{\gamma}|_{[a,b]} = \gamma$.

By theorem 3.4 we find a path of generating functions for $\tilde{\gamma}(t)$. The construction in [S] shows that these functions are in fact quadratic forms $Q_t : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}$, $(x,v) \mapsto Q_t(x,v)$ generating $\tilde{\gamma}(t)$.

Proposition 4.2 *The difference ind Q_b − ind Q_a is well defined.*

<u>Proof:</u> The proof of theorem 3.5 works in the case of linear Lagrangians instead of Lagrangians coinciding with the zero section outside a compact set. Consequently Q_t is unique up to equivalence and stabilization.

Instead of proving this and since we only need uniqeness of the difference of the indices we proof the proposition by using theorem 3.5.

We choose a Hamiltonian H such that for its flow we have $\varphi_t(\mathbb{R}^n \times \{0\}) = \gamma(t)$. We find a cut off function $\chi : \mathbb{R}^{2n} \to \mathbb{R}$ such that for a ball of radius R we have $\chi(B(R)) = 1$ and $\chi(\mathbb{R}^{2n} \setminus B(2R)) = 0$ with R so great that for the flow $\tilde{\varphi}$ of $\tilde{H} = \chi \cdot H$ we still have $\tilde{\varphi}_t(0) = \varphi_t(0)$ and $T_0\tilde{\varphi}_t = T_0\varphi_t$. Let $\tilde{S}_t$ be a family of generating functions for $\tilde{\varphi}_t(\mathbb{R}^n \times \{0\})$.

Let $\tilde{S}'$ be a second family of generating functions for $\tilde{\varphi}_t(\mathbb{R}^n \times \{0\})$. We observe that due to the uniqueness theorem the indices of $d^2\tilde{S}(x, v)$ and $d^2\tilde{S}'(x, v')$ at points which generate the same point in L only differ by the index of a quadratic form with which we have stabilized. This does not affect the difference of two indices. Consequently our definition is independent of the particular gfqi for $\tilde{\varphi}(\mathbb{R}^n \times \{0\})$ chosen.

We now look for the relation between Q_t and $\tilde{S}_t$. We can homotop from $\gamma(t)$ to $\tilde{\varphi}_t(\mathbb{R}^n \times \{0\})$ by considering $s \cdot \chi H$.

During the homotopy the parts of the generating functions generating points in $B(R)$ don't need to change. Outside $B(R)$ the generating functions change. In particular we might have to stabilize with a quadratic form Q and then compose with a fibre preserving diffeomorphism. These operations do not affect the difference of indices of critical points (x, v) with $i_S(x, v) \in B(R)$.

Consequently, given Q_t and two gfqi S^1 and S^2 corresponding to two cut off functions we have for points generating the same point in $B(R)$ that $\mathrm{ind}Q_t = \mathrm{ind}\tilde{S}^1_t - \mathrm{ind}Q^1 = \mathrm{ind}\tilde{S}^2_t - \mathrm{ind}Q^2$. This shows that our definition is independent of the cut off function chosen.

A similar argument shows that it is independent of the particular Q_t chosen: For two paths Q_t and Q'_t of generating forms we have with the same cut off function $\mathrm{ind}Q^1_t + \mathrm{ind}Q^1 = \mathrm{ind}\tilde{S}_t = \mathrm{ind}Q^2_t + \mathrm{ind}Q^2$.) $\square$

4.2 The Maslov index for a path of Lagrangian vector spaces.

In view of proposition 4.2 we define

Definition 4.3 *For a continuous and piecewise differentiable path* $\gamma : [a, b] \to \Lambda(n)$ *of Lagrangian subspaces in* $\mathbb{R}^{2n}$*the* Maslov index $\mu_V(\gamma)$ *is the integer defined as*

$$\mu_V(\gamma) = ind\ Q_b - ind\ Q_a$$

where Q_t is a path of generating forms for $\gamma(t)$.

Remarks: (i) The subscript V refers to 'Viterbo' and indicates that this is not the standard index.
(ii) $\mu_V(\gamma)$ measures with multiplicities how often γ intersects the distinguished horizontal Lagrangian plane $\mathbb{R}^n \times \{0\}$. The form Q_t is non degenerate in the fibers since $T\frac{\partial Q}{\partial v}$ has maximal rank, see definition 3.1. The index indQ_t can only change if $\operatorname{rank}(dQ) < n+k$, that is if $\operatorname{rank}(\frac{\partial Q}{\partial x}) < n$. This is equivalent to $\dim(\gamma(t) \cap \mathbb{R}^n \times \{0\}) > 0$.

The Maslov index has the following properties:

Proposition 4.4 *$\mu_V(\gamma)$ satisfies*

(i) **Affine scale invariance:**

For $k > 0$ and $l \geq 0$ consider the map $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(t) = kt + l$. Then

$$\mu_V(\gamma) = \mu_V(\gamma \circ \psi).$$

(ii) **Deformation invariance:**

If $\Gamma : (s,t) \to \Gamma(s,t)$ is a parametrized surface in $\Lambda(n)$ then

$$\mu_V(\Gamma(\cdot,1)) - \mu_V(\Gamma(\cdot,0)) = \mu_V(\Gamma(1,\cdot)) - \mu_V(\Gamma(0,\cdot)).$$

(iii) **Path additivity:**

If $\gamma : [a,c] \to \Lambda(n)$ and $c > b$ then

$$\mu_V(\gamma) = \mu_V(\gamma|_{[a,b]}) + \mu_V(\gamma|_{[b,c]})$$

(iv) **Symplectic additivity:**

Let $\gamma : [a,b] \to \Lambda(n)$ and $\delta : [a,b] \to \Lambda(m)$. For the direct sum $\gamma \oplus \delta : [a,b] \to \Lambda(n+m)$ we have

$$\mu_V(\gamma \oplus \delta) = \mu_V(\gamma) + \mu_V(\delta)$$

(v) **Normalization:**

Define $\gamma_0 : [-\pi,\pi] \to \Lambda(2)$ by the formula

$$\gamma_0(t) = \mathbb{R}\, e^{it} = \{(r\cos t, r\sin t) \mid r \in \mathbb{R}\},$$

then

$$\begin{aligned}
\mu_V(\gamma_0)|_{[-\pi/4,0]} &= -1\\
\mu_V(\gamma_0)|_{[0,\pi/4]} &= 0\\
\mu_V(\gamma_0)|_{[-\pi/4,\pi/4]} &= -1\\
\mu_V(\gamma_0) &= -2.
\end{aligned}$$

Proof: (i),(iii), (iv) are clear. To prove (ii) we consider a family $Q_{s,t}$ of generating forms for $\Gamma(s,t)$. We compute the index along the path γ_1 which is defined to be $\Gamma(s,0)$ followed by $\Gamma(1,t)$. By path additivity

$$\mathrm{ind}Q_{1,0} - \mathrm{ind}Q_{0,0} + \mathrm{ind}Q_{1,1} - \mathrm{ind}Q_{1,0} = \mathrm{ind}Q_{1,1} - \mathrm{ind}Q_{0,0}.$$

We obtain the same result by considering the path γ_2 which is $\Gamma(0,t)$ followed by $\Gamma(s,1)$. This proves (ii). As for (v) we only have to observe that a generating form for $\gamma|_{[-\pi/4,\pi/4]}$ and for $\gamma|_{[3\pi/4,5\pi/4]}$ is $Q_t(x) = \frac{1}{2}\frac{\sin t}{\cos t}x$. Since $\gamma(t)$ has intersection with $\mathbb{R}^n \times \{0\}$ only for $t = 0$ and π this yields the result. □

4.3 The Maslov index for a path of symplectic automorphisms

Consider a linear symplectic automorphism $\phi \in \mathrm{Sp}(n)$. Remember that the map τ from section 2 identifies $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, (-\omega \oplus \omega)$ with $T^*\mathbb{R}^{2n}$. Hence $\tau(\Gamma(\phi))$ is a linear Lagrangian subspace in $T^*\mathbb{R}^{2n}$ and so has a generating form Q. In the following Q is called a generating form for ϕ though Q generates, more precisely

$$\begin{aligned}
L &= \{\tau(x,\phi(x))\,|\,x \in \mathbb{R}^{2n}\}\\
&= \{\tau(I \times \phi)\tau^{-1}(x,0)\,|\,x \in \mathbb{R}^{2n}\}
\end{aligned}$$

For a path $\phi : [0,1] \to \mathrm{Sp}(n)$ we define the Maslov index

$$\mu_V(\phi) = \mu_V(\tau\Gamma(\phi)).$$

This Maslov index has the properties of affine scale invariance (i), deformation invariance (ii), path additivity (iii) and symplectic additivity

(iv) from proposition 4.4. As for the normalization we get: Consider the path of symplectic automorphisms

$$\phi^0 :]-\pi, \pi[\to \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

We compute with $x = (q, p)$ and $\phi^0_t(x) = (\bar{q}, \bar{p})$.

$$\begin{aligned} & (\frac{q+\bar{q}}{2}, \frac{p+\bar{p}}{2}, \bar{p} - p, q - \bar{q}) \\ = \; & (\frac{1}{2}(q(1+\cos t) - p\sin t), \frac{1}{2}(p(1+\cos t) + q\sin t), \\ & p(1-\cos t) + q\sin t, q(1-\cos t) + p\sin t) \end{aligned}$$

Changing the coordinates

$$z_1 = \frac{q(1+\cos t) - p\sin t}{2} \quad \text{and} \quad z_2 = \frac{p(1+\cos t) + q\sin t}{2} \tag{3}$$

we get

$$\tau\Gamma(\phi^0(t)) \;=\; \left\{ (z_1, z_2, \frac{2\sin t}{1+\cos t} z_1, \frac{2\sin t}{1+\cos t} z_2) \mid (z_1, z_2) \in \mathbb{R}^{2n} \right\}$$

whose generating form is given by

$$\frac{\sin t}{1+\cos t}(x^2 + y^2).$$

One computes:

$$\mu_V(\phi^0|_{[-\pi/4,0]}) = -2$$

$$\mu_V(\phi^0|_{[0,\pi/4]}) = 0$$

$$\mu_V(\phi^0|_{[-\pi/4,\pi/4]}) = -2$$

The Lagrangian plane $\tau\Gamma(\phi(t))$ has non trivial intersection with $\mathbb{R}^{2n} \times \{0\}$ if and only if $\phi(t)$ has an eigenvalue 1. Since the above path ϕ^0 has an eigenvalue 1 only for $t = 0$ we conclude for the whole loop

$$\mu_V(\phi^0|_{[-\pi,\pi]}) = -2.$$

We next elaborate on the relation between the eigenvalues of $\phi(t)$ and $\mu_V(\phi)$ for arbitrary loops of symplectic automorphisms.

Definition 4.5 *With $\phi \in Sp(n)$ we associate the quadratic form $q_\phi(x) := \omega(x, \phi x)$ and define the integer $ind(\phi) := ind\ q_\phi$.*

Remark: The symmetric bilinear form associated to q_ϕ is

$$\begin{aligned} b_\phi(x,y) &= 1/2(\omega(x,\phi y) + \omega(y, \phi x)) \\ &= 1/2(< Jx, \phi y > + < Jy, \phi x >) \\ &= 1/2(< x, -J\phi y > + < \phi^T Jy, x >) \end{aligned}$$

so that b_ϕ is given by $\phi^T J - J\phi$ where ϕ^T denotes the transposed map.

Proposition 4.6 *Assume $\phi \in Sp(n)$, then:*

(i) If $P \in Sp(n)$ and $\psi := P^{-1}\phi P$ then $ind\ (\psi) = ind\ \phi$

(ii) If ϕ does not have an eigenvalue -1 and $Q : \mathbb{R}^n \to \mathbb{R}$ is a generating form for ϕ then Q is conjugated to q_ϕ, i.e. there exists a matrix A such that $A^T Q A = q_\phi$. Hence $ind\ Q = ind\ \phi$.

(iii) If $\phi : [a,b] \to Sp(n)$ is a path such that $(Ker(\phi(t)^2 - I))$ is constant then $ind\ (\phi(1)) = ind\ (\phi(0))$. This is, in particular, the case if $\phi(t)$ never has the eigenvalues ± 1.

(iv) If $\phi : [a,b] \to Sp(n)$ is a path such that $\phi(t)$ never has an eigenvalue -1 then

$$\mu_V(\phi) = ind\ (\phi(1)) - ind\ (\phi(0)).$$

Proof: (i) $q_\psi(x) = \omega(x, P^{-1}\phi P x) = \omega(Px, \phi Px) = q_\phi(Px)$.

(ii) If -1 is no eigenvalue of ϕ then

$$\Gamma(\phi) \cap \{(z, -z) \mid z \in \mathbb{R}^{2n}\} = \{0\}.$$

Consequently

$$\tau(\Gamma(\phi)) \cap \{0\} \times \mathbb{R}^n = \{0\}.$$

$\tau(\Gamma(\phi))$ therefore projects well on $\mathbb{R}^n \times \{0\}$ so that we can write

$$\tau(\Gamma(\phi)) = \{(z, Az) \mid z \in \mathbb{R}^{2n}\}$$

with a symmetric matrix A. We have that $Q(z) := 1/2 < z, Az >$ is a generating form for ϕ. Defining $x = (q,p)$ and $\phi(q,p) = (\bar{q},\bar{p})$ and applying the coordinate change $z = (\frac{q+\bar{q}}{2}, \frac{p+\bar{p}}{2})$ we get that

$$< (\frac{q+\bar{q}}{2}, \frac{p+\bar{p}}{2}), (\bar{p}-p, q-\bar{q}) >$$

is conjugated to $< z, Az >$. We calculate

$$\begin{aligned} & < (\frac{q+\bar{q}}{2}, \frac{p+\bar{p}}{2}), (\bar{p}-p, q-\bar{q}) > \\ = & < \frac{q+\bar{q}}{2}, \bar{p}-p > + < \frac{p+\bar{p}}{2}, q-\bar{q} > \\ = & < q, \bar{p} > - < \bar{q}, p > \\ = & \ \omega(x, \phi x) = q_\phi(x) \end{aligned}$$

(iii) The index ind $(\phi(t))$ can only change where $\mathrm{Ker}(\phi(t)^T J - J\phi(t))$ changes dimension. We compute

$$\begin{aligned} \dim\ (\mathrm{Ker}(\phi(t)^T J - J\phi(t))) &= \dim\ (\mathrm{Ker}(\phi(t)^T J\phi(t) - J\phi(t)^2)) \\ = \dim\ (\mathrm{Ker}(J - J\phi(t)^2)) &= \dim\ (\mathrm{Ker}(-I + \phi(t)^2)) \end{aligned}$$

(iv) is a consequence of (ii). □

4.3.1 A Generic formula

We define

$$Sp_k(n) := \{\phi \in Sp(n) \,|\, \dim\ \ker(\phi - I) = k\}$$

Proposition 4.7 *Let $\phi(t) : [0,1] \to Sp(n)$ be a path such that there is only a finite number of times $0 < t_1 < \ldots < t_i < \ldots < t_N$ such that* $\det(\phi(t_i) - I) = 0$. *Assume that at these points* $\det(\phi(t_i) + I) \neq 0$. *If $\phi(1) \in Sp_{k_1}(n)$ we assume $\dot{\phi}(1), \dot{\phi}(0) \notin TSp_k(n)$ for all $k \neq 0$.*

Then it holds that

$$\begin{aligned} \mu_V(\phi) &= ind\ \phi(\epsilon) - ind\ \phi(0) + \Sigma_{k=1}^N (ind\ \phi(t_{k+\epsilon}) - ind\ \phi(t_{k-\epsilon})) \\ & \quad + ind\ \phi(1) - ind\ \phi(1-\epsilon) \end{aligned}$$

for $\epsilon > 0$ small.

Proof: By path additivity

$$\begin{aligned}\mu_V(\phi) &= \mu_V(\phi)|_{[0,\epsilon]} + \mu_V(\phi)|_{[\epsilon,t_1-\epsilon]} \\ &+ \sum_{k=1}^{N-1} \left(\mu_V(\phi)|_{[t_k-\epsilon,t_k+\epsilon]} + \mu_V(\phi)|_{[t_k+\epsilon,t_{k+1}-\epsilon]}\right) \\ &+ \mu_V(\phi)|_{[t_N-\epsilon,t_N+\epsilon]} + \mu_V(\phi)|_{[t_N+\epsilon,1-\epsilon]} + \mu_V(\phi)|_{[1-\epsilon,1]}.\end{aligned}$$

On the intervals $[\epsilon, t_1 - \epsilon]$ and $[t_i + \epsilon, t_{i+1} - \epsilon]$ and $[t_N + \epsilon, 1 - \epsilon]$ there occurs no eigenvalue -1 for $\phi(t)$. Hence these intervals don't contribute to the Maslov index. On the other intervals we use proposition 4.6, (iv) to compute the indices. □

The crucial point is that every path can be deformed to such a path in view of the next result.

Proposition 4.8 *Every path $\phi(t) : [0,1] \to Sp(n)$ can be continuously deformed into a path satisfying the hypothesis of proposition 4.7 are satisfied in such a way that the Maslov index remains constant.*

Proof: We first need some facts about the structure of $Sp(n)$. Let $\Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$ be the set of all Lagrangian subspaces in $(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, -\omega \oplus \omega)$. Define

$$\Lambda_0 = \{L \in \Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \,|\, L \cap \{0\} \times \mathbb{R}^{2n} = 0\}$$

We define a map $Sp(n) \to \Lambda_0$ by $\phi \mapsto \Gamma(\phi)$. Clearly this map is injective. It is also surjective: Denote by $\pi_1 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the projection on the first factor. Since $L \in \Lambda_0$ projects well on $\mathbb{R}^{2n} \times \{0\}$, for every $x \in \mathbb{R}^{2n}$ there is a unique vector $(x, Px) \in L$. Thus we have constituted a linear map $P : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. This map is symplectic. Indeed, since L is Lagrangian we have for every $x, y \in \mathbb{R}^{2n}$

$$0 = (-\omega \oplus \omega)((x, Px), (y, Py)) = -\omega(x, y) + \omega(Px, Py).$$

Now $Sp_k(n)$ is mapped onto

$$\Lambda_k = \{L \in \Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \,|\, \dim L \cap \Delta = k\}.$$

This is a submanifold of codimension $k(k+1)/2$ in $\Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$, see [GS], proposition IV.3.5.

This shows that for $0 < t < 1$ we can deform ϕ such that it intersects none of the $Sp_k(n)$ with $k > 1$ and $Sp_1(n)$ only transversally. If $\phi(1) \in Sp_{k_1}(n)$ we can assume $\dot{\phi}(1) \notin TSp_k(n)$ for all $k \neq 0$ and the same for $\phi(0)$.

We now prove that we can avoid the eigenvalue -1 when there is an eigenvalue 1.

Let $\phi(t) \in Sp_k(n)$. In a basis it can be written as

$$\begin{pmatrix} \begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} & 0 & \dots & 0 \\ 0 & \begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & P \end{pmatrix}$$

with $P \in Sp_0(n-k)$. As above the sets

$$Sp_l^-(n) = \{\phi \in Sp(n) \mid \dim\ \ker(\phi + I) = l\}$$

have codimension $l(l+1)/2$ in $\Lambda(\mathbb{R}^{2n} \times \mathbb{R}^{2n})$. Thus there is a $\tilde{P}$ arbitrarily closed to P without eigenvalue -1 and we can deform $\phi(t)$ such that it crosses $Sp(k)$ with $\det(\phi(t_k) + I) \neq 0$. □

4.4 The Maslov index for a periodic orbit

We consider a path $\varphi : [0,1] \to \mathrm{Symp}(n)$, $t \mapsto \varphi_t$ of (non linear) symplectomorphisms and a periodic orbit $\varphi_t(x)$. Define the Maslov index of the periodic orbit by

$$\mu_V(x) = \mu_V(d\varphi_t(x)).$$

The most important example is given by the flow φ_t of a Hamiltonian vector field. In this case $d\varphi_t(x)$ solves the linearized Hamiltonian equations:

$$\frac{d}{dt}(d\varphi_t(x)) = -JH''(\varphi_t(x)).d\varphi_t(x) \quad , \quad d\varphi_0(x) = Id.$$

Proposition 4.9 *Let φ_t be a path of symplectomorphisms with $\varphi_0 = I$. Let $S_t : \mathbb{R}^{2n} \times \mathbb{R}^k \to \mathbb{R}$ be a family of gfqi for φ_t and $z = (x, v)$ be the critical point of S_1 associated to a fixed point x of φ_1. Let Q_∞ be the quadratic form to which S_1 is equal 'at infinity'. Then*

$$\mu_V(x) = ind\ d^2S(z) - ind\ Q_\infty.$$

Proof: Define $\psi_t := \tau(I \times \varphi_t)\tau^{-1}$. Then S_t generates

$$L_t = \{\psi_t(x, 0) \,|\, x \in \mathbb{R}^{2n}\} = \tau\Gamma(\varphi_t).$$

We define the path $z(t) = \psi_t(x, 0) = \tau(x, \varphi_t(x))$. We find a path $\xi : [0, 1] \to \mathbb{R}^{2n} \times \mathbb{R}^k$ such that $\xi(t) \in \Sigma_{S_t}$ and $i_{S_t}\xi(t) = z(t)$ and $\xi(1) = (z, v)$. By proposition 4.1, $d^2S(\xi(t))$ generates

$$T_{z(t)}L_t = T_{z(t)}(\tau\Gamma(\varphi_t)) = \tau\Gamma(d\varphi_t(x)).$$

Hence by proposition 4.9

$$\mu_V(x) = \text{ind } d^2S(z(1)) - \text{ind } d^2S(z(0)).$$

In the definition of generating functions we assumed that outside a compact set S is equal to a quadratic form Q_∞. Since $\varphi_t = I$ outside a compact set Q_∞ generates the zero section. Since $\varphi_0 = I$ the function S_0 generates the zero section as well.

We choose a point x_∞ outside the support of φ and a smooth path $g(s)$ connecting $(x_\infty, 0)$ with $z(0) = (x, v)$ such that $g(s)$ is in the critical locus Σ_{S_0}. Since S_0 generates the zero section we see that $d^2S_0(g(s))$ generates the zero section. We have $\dim(\text{Ker} d^2S_0(g(s))) = \dim((\tau\Gamma(\phi)) \cap \mathbb{R}^{2n} \times \{0\}) = 2n$. Consequently, the dimension of the kernel is constant, and the index of $d^2S_0(g(s))$ does not change.

This proves

$$\text{ind } d^2S_0(x, v) = \text{ind } d^2S_0(z(0)) = \text{ind } Q_\infty$$

$\square$

5 Viterbo's capacities for strictly convex hypersurfaces

A smooth closed hypersurface Σ is called strictly convex if it has positive sectional curvature. There exists a strictly convex set U such that $\partial U = \Sigma$. For simplicity we assume that $0 \in U$.

Hofer, Wysocki and Zehnder in [HWZ1] defined a generalized Conley–Zehnder index. In [HWZ2] they considered a strictly convex energy surface Σ and a Hamiltonian function h such that $h^{-1}(1) = \Sigma$ and $h(rx) = r^2 h(x)$. They gave bounds on the index for periodic orbits associated to the Hamiltonian flow of h.

In section 5.1.2 we do the same for Viterbo's index. In section 5.1.3 we generalize this to $G \circ h$ with a smooth function G. In 5.2 we shall use these results in order to prove that $c(\Sigma) = c(U)$.

5.1 Maslov indices

5.1.1 Strictly convex hypersurfaces

Let Σ be strictly convex.

From appendix A we know that associated to Σ there is a characteristic line bundle $\mathcal{L}_\Sigma \to \Sigma$, $\mathcal{L}_\Sigma \subset T\Sigma$ defined by

$$\mathcal{L}_\Sigma = \{(x,\xi) \in T\Sigma \,|\, \omega(\xi,\eta) = 0 \text{ for all } \eta \in T_x\Sigma\}.$$

This gives an integrable distribution on Σ called the characteristic foliation. We denote by $L_\Sigma(x)$ the leaf through $x \in \Sigma$.

Since $H' - T\Sigma$ we have that $\xi = JH' \in T_\Sigma$. For $\eta \in T\Sigma$ we compute

$$\omega(JH',\eta) =< -H',\eta >= 0.$$

This shows that $JH' \in \mathcal{L}_\Sigma$. Hence the orbits of the flow associated to H are the leaves of the characteristic foliation. Furthermore

$$\lambda_x(JH') = 1/2 < Jx, JH' >= -1/2 < x, H' >\neq 0$$

since Σ is strictly convex. Consequently $\lambda|_{T_x\Sigma} \neq 0$ for all $x \in \Sigma$ and $\Xi_x = \ker \lambda_x|_\Sigma$ is a $(2n-2)$–dimensional subspace on which ω is non degenerate. So $\lambda \wedge \omega^{n-1}$ is a volume form on Σ. By definition A.1 that means that Σ is of restricted contact type.

5.1.2 The indices for Hamiltonians with H'' definite

Proposition 5.1 *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a time independent Hamiltonian with $H^{-1}(1) = \Sigma$. Let φ_t be the associated flow.For a periodic orbit $x(t) = \varphi_t(x)$ with $d\varphi_1(x) \in Sp_k(n)$ we have*

i) $\mu_V(x) \geq 2n$ *if H'' is positive definite for all $x \in \Sigma$*
ii) $\mu_V(x) \leq -k$ *if H'' is negative definite for all $x \in \Sigma$.*

Proof of part i) We consider paths $d\varphi_t(x) =: \phi(t)$ of symplectic matrices. Since

$$\dot{\phi}(t) = -JH''.\phi(t) \iff H'' = J\dot{\phi}(t)(\phi(t))^{-1}$$

the small perturbations leading to the formula in section 4.3.1 do not affect the fact that H'' is positive definite. The Taylor series of $\phi(t)$ begins with

$$\phi(t+\epsilon) \approx \phi(t) + \epsilon\dot{\phi}(t) = \phi(t) - \epsilon JH'' d\varphi_t(x) = \phi(t) - \epsilon JH''\phi(t),$$

neglecting terms of order $O(\epsilon^2)$. From $\dot{\phi}(0) \notin TSp_k(n)$ we deduce

$$\begin{aligned} \text{ind } \phi(\epsilon) - \text{ind } \phi(0) = \; &= \text{ind } (I - \epsilon JH'') - \text{ind } (I) = \text{ind } (I - \epsilon JH'') \\ &\overset{def}{=} \text{ind } ((I - \epsilon H'' J^T)J - J(I - \epsilon JH'')) \\ &= \text{ind } (-2\epsilon H'') = 2n \end{aligned}$$

We now consider the indices for $t = t_k$ with $t_k < 1$. We abbreviate $\phi(t_k) = \phi$. Due to the deformations the eigenvalue 1 has multiplicity 1 and no eigenvalue -1 occurs.

We consider the flow of the eigenvalues of the symmetric operator $T(t) = \phi(t)^T J - J\phi(t)$. Using Kato's perturbation theory, [K], we observe that there is a differentiable function $\lambda : [t_k - \epsilon, t_k + \epsilon] \to \mathbb{R}$ where $\lambda(t)$ is an eigenvalue of $T(t)$ and $\lambda(t_k) = 0$. (Theorem II.6.1 in [K]). We have that

$$\text{ind } \phi(t_k + \epsilon) - \text{ind } \phi(t_k - \epsilon) = -1 \;\text{ if } \lambda(t_k + \epsilon) > 0$$

$$\text{ind } \phi(t_k + \epsilon) - \text{ind } \phi(t_k - \epsilon) = 1 \;\text{ if } \lambda(t_k + \epsilon) < 0$$

We have to show that for $H'' > 0$ the second case occurs. The first coefficient in the Taylor expansion of λ, giving the sign of $\lambda(\epsilon)$ is given by formula 2.32 in paragraph II.2 in [K]:

$$\lambda^{(1)} = \text{tr } (T^{(1)}P) \tag{4}$$

where $T^{(1)}$ is the first element in the Taylor expansion of $T(t)$ and P is the projection onto the zero eigenspace of $T(t_k)$.

We have

$$\begin{aligned} & \phi(t_k+\epsilon)^T J - J\phi(t_k+\epsilon) \\ = \; & (\phi - \epsilon J H''\phi)^T J - J(\phi - \epsilon J H''\phi) + O(\epsilon^2) \\ = \; & \phi^T J - J\phi - \epsilon((\phi^T H'' J^T)J - JJH''\phi) + O(\epsilon^2). \end{aligned}$$

Consequently

$$T^{(1)} = -\phi^T H'' - H''\phi.$$

We choose a basis in which the matrices ϕ and P take the form

$$\phi = \begin{pmatrix} \begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix} & 0 \\ 0 & * \end{pmatrix} \qquad \text{and} \qquad P = \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix}. \tag{5}$$

We compute:

$$\begin{aligned} \text{tr } T^{(1)}P &= \text{tr } ((-\phi^T H'' - H''\phi)P) \\ &= \text{tr } (P(-\phi^T H'') - H''\phi P) \\ &= \text{tr } \left(- \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix} H'' - H'' \begin{pmatrix} \begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} & 0 \\ 0 & 0 \end{pmatrix} \right) < 0 \end{aligned}$$

since H'' is positive definite.

If 1 is an eigenvalue of $\phi(1)$ with multiplicity k, then as in proposition 4.8 it has no eigenvalue -1. Perturbation theory provides us with k functions $\lambda_j : [1-\epsilon, 1] \to \mathbb{R}$ with $\lambda_k(1) = 0$ and k functions $\xi_j : [1-\epsilon, 1] \to \mathbb{R}^{2n}$ satisfying $T(t)\xi_j(t) = \lambda_j(t)\xi_j(t)$. We may assume that all the $\xi_j(t)$ are different for $t \neq 1$.

Formula 4 is valid for an eigenvalue of multiplicity 1. Here we have to use the reduction process of paragraph *II*.2.3 in Kato, [K]. By formula II.2.40,

$$\lambda_j(1-\epsilon) = 0 - \epsilon\lambda_j^{(1)} + O(1+\delta),$$

with $\delta > 0$. The coefficient $\lambda_j^{(1)}$ is given by $P_j T^{(1)} P_j$ where P_j is the projection onto $\mathbb{R}\cdot\xi_j(1)$. We choose a basis as in formula 5 and conclude that $\lambda_j(1-\epsilon) < 0$ and hence

$$\text{ind } \phi(1) - \text{ind } \phi(1-\epsilon) = 0.$$

Proof of part ii) Considering the spectral flow as in i) we find

$$\begin{aligned} \text{ind } \phi(\epsilon) - \text{ind } \phi(0) &= 0 \\ \text{ind } \phi(t_k+\epsilon) - \text{ind } \phi(t_k-\epsilon) &= -1 \\ \text{ind } \phi(1) - \text{ind } \phi(1-\epsilon) &= -k. \end{aligned}$$

□

5.1.3 More indices

We can also compute the index for periodic orbits associated to Hamiltonians which are not definite but of the form $H = g(\sqrt{h})$ where h is the square of the Minkowsky functional $m(x) = \inf\{\lambda \,|\, x \in \lambda U\}$. To this end we need some preparations.

Define $h = m^2$ then $h|_\Sigma = 1$ and $h(rx) = r^2 h(x)$. For $x \neq 0$ we have $h''(x) > aI$ with $a > 0$.

Differentiating $h(rx) = r^2 h(x)$ with respect to r one sees that

$$< h'(rx), x >= 2rh(x) \iff < h'(rx), rx >= 2r^2 h(x) = 2h(rx).$$

Consequently,

$$< h'(x), x >= 2h(x).$$

Differentiating this equation with respect to x one sees that

$$< h''(x)\cdot, x > + < h'(x), . >= 2 < h'(x), \cdot > .$$

We have

$$h''(x).x = h'(x).$$

We now consider the Hamiltonian function $H(x) = g(\sqrt{h(x)}) = g(m(x))$ where $g : \mathbb{R} \to \mathbb{R}$ is a C^2–function with $g(1) = 1$ and $g'(1) \neq 0$. For $x \in \Sigma = H^{-1}(1) = h^{-1}(1)$, H satisfies the equations

$$< x, H' >=< x, g'\frac{1}{2\sqrt{h}}h' >= g'(1) \tag{6}$$

and

$$H'' = g''\frac{1}{4h} < h', \cdot >< h', \cdot > +g'(-\frac{1}{4h^{3/2}}) < h', \cdot >< h', \cdot > +g'\frac{1}{2\sqrt{h}}h'' \tag{7}$$

and, since $h = 1$

$$H'' = g''\frac{1}{4} < h', \cdot >< h', \cdot > +g'(-\frac{1}{4}) < h', \cdot >< h', \cdot > +g'\frac{1}{2}h'' \tag{8}$$

$$H''.x = g''\frac{1}{2\sqrt{h}} < h', \cdot > . \tag{9}$$

Define the vector field $X(x) := -\frac{1}{g'(1)}JH'(x)$, then $X(x) \in \mathcal{L}_\Sigma$, the characteristic line bundle, defined in appendix A.

Proposition 5.2 *With $\Xi_x = \ker \lambda_x|_\Sigma$ and the above vector field $X(x)$ we conclude that*

$$span(X(x), x) \oplus \Xi_x$$

is a symplectic orthogonal splitting of $T_x\mathbb{R}^{2n}$ which is invariant under $d\varphi_t(x)$.

Proof:

$$\begin{aligned} \omega(X(x), x) &= < J(-\frac{1}{g'(1)}JH'(x)), x > \\ &= \frac{1}{g'(1)} < H'(x), x > \\ &= 1 \text{ (by equation 6).} \end{aligned}$$

For $\xi \in \Xi$ we have

$$\omega(x, \xi) = \lambda_x(\xi) = 0,$$

and since $H' - T_x\Sigma$

$$\omega(X(x), \xi) = -\frac{1}{2g'(1)} < H'(x), \xi >= 0.$$

This shows that we have a symplectic orthogonal splitting. It remains to show the invariance under $d\varphi_t(x)$. Since φ_t is symplectic we have $\omega(d\varphi_t \cdot, d\varphi_t \cdot) = \omega(\cdot, \cdot)$. Therefore, $\mathcal{L}_\Sigma$ and hence $X(x)$ is invariant under $d\varphi_t(x)$.

We want to show

$$\xi_x \in \Xi_x \Rightarrow d\varphi_t(x)\xi \in \Xi_{\varphi_t(x)}$$

which is equivalent to

$$\omega(\xi, \eta) = 0 \text{ for all } \eta \in T_x\Sigma \Rightarrow \omega(d\varphi_t(x)\xi, \eta) = 0 \text{ for all } \eta \in T_{\varphi_t(x)}\Sigma.$$

Since $\varphi_t^*\omega = \omega$ we only need to show that

$$\eta \in T_x\Sigma \iff d\varphi_t(x)\eta \in T_{\varphi_t(x)}\Sigma$$

which holds since $\varphi_t|_\Sigma : \Sigma \to \Sigma$ is a diffeomorphism.

It is left to show that $d\varphi_t(x).x \in \text{span}(\varphi_t(x), X(\varphi_t(x)))$. Since we have the symplectic orthogonal splitting of $T_{\varphi_t(x)}\mathbb{R}^{2n}$ it is enough to show that

$$\begin{aligned} & \omega(d\varphi_t(x).x, \tilde{\xi}) = 0 \text{ for all } \tilde{\xi} \in \Xi_{\varphi_t(x)} \\ \iff & \omega(x, d\varphi_t^{-1}(x).\tilde{\xi}) = 0 \text{ for all } \tilde{\xi} \in \Xi_{\varphi_t(x)} \\ \iff & \omega(x, \xi) = \lambda_x(\xi) = 0 \text{ for all } \xi \in \Xi_x \end{aligned}$$

□

In the special case $H(x) = h(x)$ we have

$$\dot{\varphi}_t(x) = -JH'(\varphi_t(x)) = -JH''(\varphi_t(x)).\varphi_t(x)$$

We remember that $d\varphi_t(x).x$ is a solution of the linearized Hamiltonian equations

$$\frac{d}{dt}(d\varphi_t(x)).x = -JH''(\varphi_t(x)).d\varphi_t(x).x \quad , \quad d\varphi_0(x).x = x.$$

We see that $\varphi_t(x)$ and $d\varphi_t(x).x$ solve the same differential equation. Since $\varphi_0(x) = x = d\varphi_0(x).x$, we have $\varphi_t(x) = d\varphi_t(x).x$

We are now ready to compute the indices.

Proposition 5.3 *Let $g : \mathbb{R} \to \mathbb{R}$ be a function with $g(1) = 1$. Let φ_t be the flow associated to the Hamiltonian $H = g(\sqrt{h})$. For a 1-periodic orbit $x(t) = \varphi_t(x)$ of the flow with $d\varphi_1(x) \in Sp_k(n)$ we have*

i)	$\mu_V(x) \geq 2n$	*if*	$g'(1) > 0$	*and*	$g''(1) > 0$
ii)	$\mu_V(x) \leq -k$	*if*	$g'(1) < 0$	*and*	$g''(1) < 0$
iii)	$\mu_V(x) \geq 2n-1$	*if*	$g'(1) > 0$	*and*	$g''(1) \leq 0$
iv)	$\mu_V(x) \leq -k+1$	*if*	$g'(1) < 0$	*and*	$g''(1) \geq 0$

<u>Proof:</u> Parts i) and ii) are special cases of proposition 5.1.

At first we observe that if we consider the Hamiltonian $cH(x)$ with $c > 0$ we get the same periodic orbits. If x is a fixed point of φ_1 then $\varphi_{ct}(x)$ is a periodic orbit of the flow associated to $cH(x)$ and the corresponding path of symplectic matrices is $d\varphi_{ct}(x)$. So the Maslov index of the orbit is the same. Consequently, by rescaling, we may only consider Hamiltonians $g(\sqrt{h})$ with $|g'(1)| = 2$ which will turn out to be convenient in view of equation (8).

Proof of part iii)

We first consider the case $n = 1$ whose proof is different from the case $n > 1$. Each matrix in $Sp(1)$ can be uniquely written as $\Phi = UP$ where $U \in SO(2)$ and

$$P \in \Pi := \{P \in Sp(1) \,|\, P \text{ is symmetric and positive definite }\}.$$

Since Π is a contractible topological space the Maslov index of a path $\phi(t)$ is determined by the $SO(2)$ factor: $\mu_V(\phi(t)) = \mu_V(U(t))$. In polar coordinates we have with positive functions f_1 and f_2:

$$H(x, y) = g(\sqrt{h(x, y)}) = f_1(r) \cdot f_2(\theta) = \tilde{H}(r, \theta).$$

Since $g' > 0$ we have $f_1'(r) > 0$. Hence

$$\dot{\theta} = -\frac{\partial \tilde{H}}{\partial r} \cdot \frac{1}{r} = -\frac{f_1'(r) f_2(\theta)}{r} < 0.$$

Hence every vector in $\mathbb{R}^2$ turns in negative direction under the flow $\varphi(t)$. This is also valid for $d\varphi_t(x) = \phi(t)$. Therefore every vector turns in negative direction under $U(t)$. Consequently, after rescaling, $U(t)$ is the flow of a Hamiltonian of the form $c(x^2 + y^2)$. This gives $2n \geq \mu_V(U(t)) = \mu_V(\phi(t))$.

We compare the case $n > 1$ with $H = h$ by using the splitting of proposition 5.1. Instead of the index of $\phi(t)$ we will compute the index in a trivialization

$$\Gamma_t \phi(t) \Gamma_0^{-1} : \mathrm{span}(X(x(0)), x(0)) \oplus \Xi_{x(0)} \to \mathrm{span}(X(x(t)), x(t)) \oplus \Xi_{x(t)}.$$

The construction of the trivialization is as follows: For $n > 1$ every loop in Σ is contractible we find a map $u : [0,1]^2 \to \Sigma$ such that $u(1,t) = x(t)$ and $u(s,0) = u(s,1) = u(0,t) = u(1,0)$. We trivialize Ξ_x and $\mathrm{span}(X(x), x)$ over the image of u. Via u we pull back this trivialization to $[0,1]^2$ and get a map

$$\Gamma : [0,1]^2 \to Sp(n)$$

with

$$\Gamma(s,t) : T_{u(s,t)}\mathbb{R}^{2n} \to \mathrm{span}(X(u(s,t)), u(s,t)) \oplus \Xi_{u(s,t)}$$

and $\Gamma(s,0) = \Gamma(s,1) = \Gamma(0,t) = \Gamma_0$

We have $\mu_V(\phi(t)) = \mu_V(\Gamma_0 \phi(t) \Gamma_0^{-1})$. In fact, since $Sp(n)$ is path connected we can join Γ_0 with I by a path $\psi(r) \in Sp(n)$. Using the deformation invariance for $\psi(r)\phi(t)\psi^{-1}(r)$ we get

$$\begin{aligned}
& \mu_V(\phi(t)) - \mu_V(\Gamma_0 \phi(t) \Gamma_0^{-1}) \\
= \; & \mu_V(\psi(1)\phi(t)\psi(1)^{-1}) - \mu_V(\psi(0)\phi(t)\psi(0)^{-1}) \\
= \; & \mu_V(\psi(r)\phi(1)\psi^{-1}(r)) - \mu_V(\psi(r)\phi(0)\psi(r)^{-1}) \\
= \; & \mu_V(\psi(r)\phi(1)\psi(r)^{-1}) - 0 = 0.
\end{aligned}$$

We have used that $\phi(0)$=I=$\psi(1)$ and that $\mu_V(\psi(r)\phi(1)\psi(r)^{-1})$ can only be non zero if

$$\mathrm{Ker}(\psi(r)\phi(1)\psi(r)^{-1} - I)$$

changes dimension. But

$$\dim(\mathrm{Ker}(\psi(r)\phi(1)\psi(r)^{-1} - I)) = \dim(\mathrm{Ker}(\phi(1) - I))$$

which is independent of r.

Using the deformation invariance again we find

$$\begin{aligned}
& \mu_V(\Gamma(1,t)\phi(t)\Gamma_0^{-1}) - \mu_V(\phi(t)) \\
= \; & \mu_V(\Gamma(1,t)\phi(t)\Gamma_0^{-1}) - \mu_V(\Gamma_0\phi(t)\Gamma_0^{-1}) \\
= \; & \mu_V(\Gamma(1,t)\phi(t)\Gamma_0^{-1}) - \mu_V(\Gamma(0,t)\phi(t)\Gamma_0^{-1}) \\
= \; & \mu_V(\Gamma(s,1)\phi(1)\Gamma_0^{-1}) - \mu_V(\Gamma(s,0)\phi(0)\Gamma_0^{-1}) \\
= \; & 0,
\end{aligned}$$

since these are constant paths.

Writing $\Gamma(1,t) = \Gamma_t$ we have to compute the index of the path of matrices

$$\Gamma_t\phi(t)\Gamma_0^{-1} : \mathrm{span}(X(x(0)), x(0)) \oplus \Xi_{x(0)} \to \mathrm{span}(X(x(t)), x(t)) \oplus \Xi_{x(t)}$$

which are matrices of the form

$$\begin{pmatrix} \phi_1(t) & 0 \\ 0 & \phi_2(t) \end{pmatrix}.$$

Here $\phi_1(t)$is a 2×2 matrix and $\phi_2(t)$ is a $(2n-2)\times(2n-2)$ matrix. Since $\phi(t)X(x(0)) = X(x(t))$ and since $\phi_1(t)$ is symplectic

$$\phi_1(t) = \begin{pmatrix} 1 & 0 \\ \alpha(t) & 1 \end{pmatrix}.$$

This matrix never has an eigenvalue -1, so that $\mu_V(\phi_1(t)) = \mathrm{ind}\ \phi_1(1) - \mathrm{ind}\ \phi_1(0)$. We compute

$$\phi_1(t)^T J - J\phi_1(t) = \begin{pmatrix} 2\alpha(t) & 0 \\ 0 & 0 \end{pmatrix}.$$

We conclude that $\mu_V(\phi_1(t)) = 0$ or 1. Differentiating the flow we find by equation (8)

$$\begin{aligned} \frac{d}{dt}(\phi(t)) &= -JH''\phi(t) \\ &= -J((g''-g')(\frac{1}{4} < h', \cdot >< h', \cdot >) + g'\frac{1}{2}h'')\phi(t). \end{aligned}$$

Since $g' = 2$ we see that restricted to Ξ_x (where $< h', \cdot >= 0$) we have: $H''|_{\Xi_x} = h''|_{\Xi_x}$. If $\phi^h(t)$ denotes the linearized time one flow associated to h'' along $x(t)$ we have for

$$\Gamma_t \phi^h(t) \Gamma_0^{-1} = \begin{pmatrix} \phi_1^h(t) & 0 \\ 0 & \phi_2^h(t) \end{pmatrix}$$

that the matrices $\phi_2^h(t)$ and $\phi_2(t)$ corresponding to the Ξ_x component agree. Consequently,

$$\begin{aligned} 2n \le \mu_V(\phi^h(t)) &= \mu_V(\phi_1^h(t)) + \mu_V(\phi_2^h(t)) \\ &= \mu_V(\phi_1^h(t)) - \mu_V(\phi_1(t)) + \mu_V(\phi(t)). \end{aligned}$$

Since for the flow associated to h we have $d\phi_t(x).x = \phi_t(x)$ and

$$d\phi_t(x(0)).X(x(0)) = X(\phi_t(x(0)))$$

we have $\phi_1(t) = I$ and $\mu_V(\phi_1(t)) = 0$. Consequently,

$$2n - 1 \le \mu_V(\phi(t)).$$

Proof of part iv) Denote by $\phi^{(-h)}$ the linearized flow associated to the Hamiltonian $-h$. Writing

$$\Gamma_t \phi^{(-h)}(t) \Gamma_0^{-1} = \begin{pmatrix} \phi_1^{(-h)}(t) & 0 \\ 0 & \phi_2^{(-h)}(t) \end{pmatrix}$$

we get since for the flow associated to $-h$ we have $d\phi_t(x(0)).X(x(0)) = X(\phi_t(x(0)))$

$$\phi_1^{(-h)}(t) = \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}.$$

Hence

$$\mu_V(\phi(t)) \le -k + 1.$$

□

5.2 The capacity

Theorem 5.4 *Let $\Sigma \subset \mathbb{R}^{2n}$ be strictly convex, bounding the open and bounded set U such that $\partial U = \Sigma$. Then*

$$c(\Sigma) = c(U).$$

Proof: For simplicity we assume that $0 \in U$.

We find a sequence H_k of Hamiltonians such that for the associated time one maps φ^k we have $c_+(\varphi^k) \to c_+(U)$. We can choose the H_k such that H_k =const= c_k for $x \in (1 - 1/k)U$ and $\mathrm{supp}(H_k) \subset U$ and $H_k \leq H_{k+1}$, see figure 3.

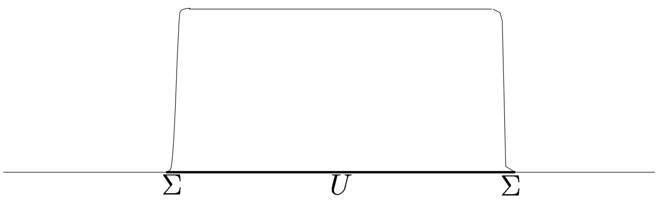

Figure 3: H_k

For fixed k and $\lambda \geq 0$ we define

$$\tilde{H}_k(x) = (1 - 1/k)^{-2} H_k((1 - 1/k)x)$$

(see figure 4) with associated time one maps $\tilde{\varphi}^k$.

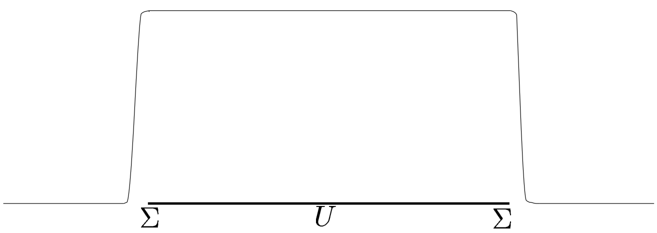

Figure 4: $\tilde{H}_k$

By proposition 3.24 we get $c_+(\tilde{\varphi}^k) = (1-1/k)^{-2}c_+(\varphi^k)$. Consequently

$$\lim_{k\to\infty} c_+(\tilde{\varphi}^k) = c(U).$$

Observe that $\tilde{H}_k$ =const on U.

For fixed k we consider a continuous family of differentiable functions $g_{k,s} : [0,1] \to \mathbb{R}$ with $s \in [0,1]$ with

$$g_{k,s}(r) = (1-s)c_k \text{ for } r = 0$$

$$g'_{k,s}(r) > 0 \text{ for } r \in]0, 1-1/(k)[$$

$$g_{k,s}(r) = c_k \text{ for } r \in [1-1/(k), 1].$$

We consider the Hamiltonians (see figure 5)

$$\tilde{H}_{k,s} = \begin{cases} \tilde{H}_k \text{ on } \mathbb{R}^{2n} \setminus U \\ g_{k,s}(\sqrt{h}) \text{ on } U. \end{cases}$$

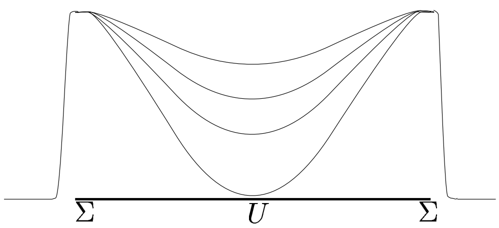

Figure 5: $\tilde{H}_{k,s}$

Let $\tilde{\varphi}_t^{k,s}$ be the flow associated to $\tilde{H}_{k,s}$. We want to show that

$$c_+(\tilde{\varphi}^k) = c_+(\tilde{\varphi}^{k,0}) = c_+(\tilde{\varphi}^{k,1}).$$

Let $\tilde{S}_{k,s} : S^{2n} \times \mathbb{R}^k \to \mathbb{R}$ be a family of generating functions generating $\tilde{\varphi}^{k,s}$. Denote by $\tilde{Q}_{k,s}$ the quadratic form at 'infinity'. By a small perturbation in a neighbourhood of the points associated to U we may assume that the critical points of the perturbed family $\hat{S}_{k,s}$ are non degenerate for almost all s whenever this critical point generates a point in U.

If $\hat{\varphi}^{k,s}$ denotes the family of symplectic diffeomorphisms associated to $\hat{S}_{k,s}$ we have that $c_+(\hat{\varphi}^{k,s})$ is arbitrary close to $c_+(\tilde{\varphi}^{k,s})$. So we are done if we show that

$$c_+(\hat{\varphi}^{k,s}) = c_+(\hat{\varphi}^{k,0}).$$

The capacity $c_+(\hat{\varphi}^{k,s})$ depends continuously on s. Since on $\mathbb{R}^{2n} \setminus U$ we have $\varphi^k = \hat{\varphi}^{k,s}$ and since the action spectrum is compact and nowhere dense we see that the fixed points x_s of $\hat{\varphi}^{k,s}$ which are associated to a critical point p_s of $\hat{S}_{k,s}$ such that

$$\hat{S}_{k,s}(p_s) = c_+(\hat{\varphi}^{k,s})$$

have to be inside U whenever the capacity changes. We can now find a contradiction to the assumption that the capacity changes by looking at the Maslov indices.

Around $x = 0$ we can assume that the Hamiltonian associated to $\tilde{\varphi}^{k,s}$ and $\hat{\varphi}^{k,s}$ is given by $c \cdot (x^2 + y^2)$ with $0 \leq c \leq \pi$. We conclude $\mu_V(V) \geq 2n$.

For a fixed point $x_s \neq 0$ of $\hat{\varphi}^{k,s}$ inside U we have that $T_{x_s}\hat{\varphi}_t^{k,s}$ is arbitrary close to $T_{x_s}\tilde{\varphi}_t^{k,s}$. By proposition 5.3 we have $\mu_V(T_{x_s}\tilde{\varphi}_t^{k,s}) \geq 2n - 1$.

We claim that $\mu_V(T_{x_s}\hat{\varphi}_t^{k,s}) \geq 2n - 1$. To this end we consider a path γ from $T_{x_s}\tilde{\varphi}_1^{k,s} \in Sp_k(n)$ to $T_{x_s}\hat{\varphi}_1^{k,s}$. Generically $T_{x_s}\hat{\varphi}_1^{k,s}$ has no eigenvalue ± 1. By the argument in proposition 4.8 we can assume that $T_{x_s}\tilde{\varphi}_1^{k,s}$ has no eigenvalue -1 either. Thus $\mu_V(\gamma) = \operatorname{ind} T_{x_s}\hat{\varphi}_1^{k,s} - T_{x_s}\tilde{\varphi}_1^{k,s} \geq 0$. By deformation invariance we have

$$\mu_V(\operatorname{ind} T_{x_s}\hat{\varphi}_t^{k,s}) = \mu_V(\operatorname{ind} T_{x_s}\tilde{\varphi}_t^{k,s}) + \mu_V(\gamma) \geq 2n - 1.$$

Hence, by proposition 4.9, $\operatorname{ind} d^2\hat{S}_{k,s}(p_s) - \operatorname{ind} \tilde{Q}_{k,s} \geq 2n - 1$. Since for almost all s we have that $d^2\hat{S}_{k,s}(p_s)$ is non degenerate we have by proposition 3.6 $\operatorname{ind} d^2\hat{S}_{k,s}(p_s) - \operatorname{ind} \tilde{Q}_{k,s} = 0$ which is a contradiction. So x_s has to be inside U and $c_+(\tilde{\varphi}^{k,s})$ cannot change.

To finish our proof we define a homotopy $\bar{g}_{k,r}$ of functions which deform $g_{k,1}$ to 0 on $[0, 1 - 2/k]$ such that $\bar{g}'_{k,r} > 0$ where $\bar{g}_{k,r} > 0$.

Define (see figure 6)

$$\bar{H}_{k,r} = \begin{cases} \tilde{H}_k \text{ on } \mathbb{R}^{2n} \setminus U \\ \bar{g}_{k,r}(\sqrt{h}) \text{ on } U. \end{cases}$$

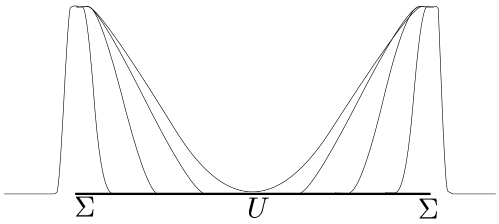

Figure 6: $\bar{H}_{k,r}$

Denote the associated flow by $\bar{\varphi}^{k,r}$. In view of the above index argument the critical point associated to $c_+(\bar{\varphi}^{k,r})$ cannot be located where $\bar{g}'_{k,r} > 0$. It also cannot be located where $\bar{g}_{k,r} = 0$ since then $c_+(\bar{\varphi}^{k,r})$ would have to jump to zero. Hence

$$c_+(\bar{\varphi}^{k,1}) = c_+(\tilde{\varphi}^{k,0}) = c_+(\tilde{\varphi}^{k}).$$

Consequently the maps $\bar{\varphi}^{k,1}$ constitute a sequence with $supp(\bar{H}_{k,1}) \longrightarrow \Sigma$ and $c_+(\bar{\varphi}^{k,1}) \longrightarrow c(U)$ $\square$

6 A 'generalized fixed point theorem'

Theorem 6.1 *Let $\Sigma \subset \mathbb{R}^{2n}$ be a closed strictly convex smooth hypersurface. Denote by $L_\Sigma(x) \subset \Sigma$ the leaf of the characteristic foliation defined by $\ker(\omega|_\Sigma)$. Let ψ be the time one diffeomorphism of the flow of a time dependent Hamiltonian vector field X_K whose Hamiltonian K is compactly supported.*

If

$$\gamma(\psi) < c(\Sigma)$$

then there exists a point $x \in \Sigma$ with $\psi(x) \in L_\Sigma(x)$.

Remarks: In Moser's theorem 2.1 one needed that K was C^1-closed to the identity. Our theorem might be interpreted as a smallness condition, too. We know that $\gamma(\psi) = 0 \iff \psi = I$ and the capacities are continuous for the C^0-norms on Hamiltonian functions and on symplectic diffeomorphisms with compact support. Every strictly convex open set contains a ball with non zero capacity. Hence the capacity of Σ is non zero. We conclude that if K is C^0-closed to zero respectively ψ is C^0-closed to the identity then a point $x \in \Sigma$ with $\psi(x) \in L_\Sigma(x)$ always exists. Thus we have replaced C^1-closed by C^0-closed.

Our theorem is not only a smallness condition. In applications one may use that $\gamma(\psi) < \sup K - \inf K$ by proposition 3.17. We do this after proving the theorem.

Proof:

In the first part the proof follows the lines of theorem 5.4.

We start with a Hamiltonian function H with $suppH \subset U$ with associated flow φ such that

$$\gamma(\psi) < c_+(\varphi) \leq c(U).$$

and $H = const = c$ on $(1-\rho)U$. We can find a family of Hamiltonian functions H_δ, $\delta \in (0, 1/2)$ such that $H_{\delta_1} \geq H_{\delta_2} > H$ for $\delta_1 < \delta_2$, $suppH_\delta \subset U$ and $H_\delta = c$ on $(1+\delta)^{-1}U$.

By proposition 3.24 we have that $\tilde{H}_\delta := (1+\delta)^2 H_\delta((1+\delta)^{-1}x)$ is a family of Hamiltonian functions such that for the associated flows $\tilde{\varphi}^\delta$ we have

$$\gamma(\psi) < c_+(\tilde{\varphi}^\delta) \leq (1+\delta)^2 c(U) < 3c(U).$$

Note that $\tilde{H}_\delta = (1+\delta)^2 c < 3c$ on U and $supp\tilde{H}_\delta \subset (1+\delta)U$. As in theorem 5.4 we can deform these Hamiltonians to Hamiltonian functions $\bar{H}_\delta$ with $supp\bar{H}_\delta \subset (1+\delta)U \setminus (1-\delta)U$ such that for the associated flows we have

$$c_+(\bar{\varphi}^\delta) = c_+(\tilde{\varphi}^\delta).$$

We have by proposition 3.13 (vi)

$$c_+(\bar{\varphi}^\delta\psi) \geq c_+(\bar{\varphi}^\delta) + c_-(\psi) = c_+(\bar{\varphi}^\delta) - \gamma(\psi) + c_+(\psi) > c_+(\psi).$$

Let $s \mapsto \bar{\varphi}^\delta_s$ be the flow of $\bar{H}_\delta$. Consider the Hamiltonian diffeomorphism $\bar{\varphi}^\delta_s\psi$. Let $H_{s,\delta}$ be a Hamiltonian associated to $\bar{\varphi}^\delta_s\psi$.

$c_+(\bar{\varphi}^\delta_s\psi)$ changes continuously in s by Proposition 3.15. The action of periodic orbits x of $\bar{\varphi}^\delta_s\psi$ with $x(0) \notin (1+\delta)U \setminus (1-\delta)U$ does not change in s. The action spectrum of $\bar{\varphi}^\delta_s\psi$ is compact and nowhere dense, so the fixed points transporting $c_+(\psi)$ to $c_+(\bar{\varphi}^\delta_1\psi)$ have to be in $(1+\delta)U\setminus(1-\delta)U$.

Therefore there has to be a $0 < s \leq 1$ and a 1–periodic orbit x associated to $H_{s,\delta}$ with $x_\delta(0) \in (1+\delta)U \setminus (1-\delta)U$ such that

$$c_+(\bar{\varphi}^\delta_s\psi) = -\mathcal{A}_H(x_\delta) = c_+(\bar{\varphi}^\delta_1\psi).$$

(Remark: we cannot take $s = 1$ because the periodic orbit representing c_+ might switch to some other region for $s < 1$ and stay there unchanged until $s = 1$.)

Define $\hat{H}_\delta := s \cdot \bar{H}_\delta$ with time one diffeomorphism $\hat{\varphi}^\delta$. Consequently $\bar{\varphi}^\delta_s\psi = \hat{\varphi}^\delta\psi$.

For each δ we have found a fixed point x^0_δ of $\hat{\varphi}^\delta\psi$ with

$$x_\delta \in \Sigma_{1+\epsilon}, \quad -\delta < \epsilon < \delta.$$

Associated to x_δ there is a curve $x_\delta(t)$, $t \in [0,2]$ with $x_\delta(t) = \psi_t(x_\delta)$, $t \in [0,1]$ and $x_\delta(t) = \hat{\varphi}^\delta_{t-1}(x_\delta)$, $t \in [1,2]$.

Denote by $L_{1+\epsilon}(x) := L_{(1+\epsilon)\Sigma}(x)$ the leaf of the characteristic foliation on $(1+\epsilon)\Sigma$ through x (and analogously with the characteristic line bundle: $\mathcal{L}_{1+\epsilon}$).

We have $x_\delta(t) \in L_{1+\epsilon}(\psi(x_\delta))$ for $t \in [1,2]$ because $(1+\epsilon)\Sigma$ is a regular energy surface for $\hat{H}_\delta$.

Let K be a Hamiltonian for ψ. We have

$$\begin{array}{rcrcrcr} \int_1^2 x_\delta^*\lambda & = & \int_0^2 x_\delta^*\lambda & + & \int_0^1 K(x_\delta(t),t)\,dt & + & \int_1^2 \hat{H}_\delta(x_\delta(t))\,dt \\ & - & \int_0^1 x_\delta^*\lambda & - & \int_0^1 K(x_\delta(t),t)\,dt & - & \int_1^2 \hat{H}_\delta(x_\delta(t))\,dt \end{array}$$

Remembering that $d\lambda = -d(pdq)$ we observe that the first three summands are equal to

$$c_+(\hat{\varphi}^\delta\psi) \le c_+(\hat{\varphi}^\delta) + c_+(\psi) \le c((1+\delta)U) + c(U) \le 4c(U).$$

The next two summands are bounded because K and K' are bounded and the last one is bounded by k.

So $\left|\int_1^2 x_\delta^*\lambda\right| \le const$. For $\delta < 1/2$ and $(x,\xi) \in \mathcal{L}_{1+\epsilon}(x)$ we have $|\lambda(x)(\xi)| \ge d|\xi|$ where $d > 0$ (because $(1+\delta)\bar{U} \setminus (1-\delta)U$ is compact and $\lambda(x)(\xi) > 0$). Since $\dot{x}_\delta(t) \in \mathcal{L}_{1+\epsilon}(x_\delta)$ for $t \in [1,2]$, $x_\delta^*\lambda$ does not change sign on $[1,2]$ and we obtain

$$const \ge \left|\int_1^2 x_\delta^*\lambda\right| = \int_1^2 |x_\delta^*\lambda| \ge \int_1^2 d\,|\dot{x}_\delta(t)|\,dt = d\cdot length(x_\delta|_{[1,2]}).$$

So $length(x_\delta|_{[1,2]}) \le l$.

By 'Arzela–Ascoli' we find a subsequence x_{δ_m} converging to a curve $x : [1,2] \to \Sigma$ with $x_{\delta_m} \to x(0)$ and $\psi(x_{\delta_m}) \to \psi(x(0))$, where

$$\psi(x_{\delta_m}) \in L_{1+\epsilon}(x_{\delta_m}).$$

The foliations on the hypersurfaces depend smoothly on the radius and the length of the leaves connecting x_{δ_m} to $\psi(x_{\delta_m})$ is bounded by l. Therefore x and $\psi(x)$ are connected by a leaf of bounded length.

This proves the theorem. □

Remark: There is some hope to generalize this theorem to starshaped hypersurfaces or even hypersurfaces of restricted contact type.

We might try to generalize the proof to coisotropic submanifolds $A \subset \mathbb{R}^{2n}$ of codimension $k > 1$. To foliate neighbourhoods of A with conformal copies of A we would need k one forms $\lambda_1, \ldots, \lambda_k$ such that $d\lambda_i = \omega$ and $\lambda_1 \wedge \ldots \wedge \lambda_k \wedge \omega^{n-k}$ is a volume form on A. Bolle in [Bol] considered such submanifolds of 'p–contact–type' where the λ_i are defined only in a neighbourhood of A.

However, to perform our estimates we need that the forms are globally defined — unfortunately this is not possible. To see this we assume that the λ_i exist globally. We then have $d\lambda_i - d\lambda_j = 0$ which implies $\lambda_i - \lambda_j = df$. We have that $d(f|_A)(x) = 0$ for some $x \in A$. Hence $\lambda_i|_A(x) = \lambda_j|_A(x)$ and $\lambda_1 \wedge \ldots \wedge \lambda_k \wedge \omega^{n-k}$ cannot be a volume form on A.

Application to harmonic oscillators

Define $H_0(q,p) = \frac{1}{2}\Sigma a_i(q_i^2 + p_i^2)$ where $a_i > 0$. Let $a := \min\{a_i \mid 1 \leq i \leq n\}$. The ellipsoid

$$\Sigma := \{(q,p) \in \mathbb{R}^{2n} \mid H_0(q,p) = 1\}$$

has capacity $c(\Sigma) = a^2\pi$: We may assume that $a = a_1$. Then $B^{2n}(a) \subset \Sigma \subset Z^{2n}(a)$ and $c(B^{2n}(a)) = a^2\pi = c(Z^{2n}(a))$.

Corollary 6.2 *Let $H_0(q,p) = \frac{1}{2}\Sigma a_i(q_i^2 + p_i^2)$ with $0 < a := \min\{a_i\}$ with associated time one diffeomorphism φ^0 and $H_1 : \mathbb{R}\times\mathbb{R}^{2n} \to \mathbb{R}$ with support in $[0,1] \times U$ with U bounded. Let φ^1 be the flow associated to H_1 and ψ be the flow of $H_0 + H_1$. If*

$$i)\ \gamma(\psi) < \pi \quad or \quad ii)\ \sup H_1 - \inf H_1 < \pi$$

then there exists a point $x \in H_0^{-1}(1) =: \Sigma$ such that $\psi(x)$ is on the same orbit of φ^0 as x.

Proof: The leaves of the characteristic foliation on Σ are the orbits of φ_t^0. Consequently, we look for points x such that $\psi(x) \in L_\Sigma(x)$. Hence i) is a simple consequence of theorem 6.1. To prove ii) we observe that

$$\psi(x) \in L_\Sigma(x) \iff (\varphi^0)^{-1}\psi(x) \in L_\Sigma(x).$$

As proved in [HZ], proposition 5.1, we see that

$$K(t,x) := (H_0 + H_1)(t, \varphi_t^0(x)) - H_0(\varphi_t^0(x)) = H_1(t, \varphi_t^0(x))$$

is a Hamiltonian for $(\varphi^0)^{-1}\psi$. Since $\sup K - \inf K = \sup H_1 - \inf H_1 < a^2\pi$ we have by proposition 3.17 $\gamma((\varphi^0)^{-1}\psi) < a^2\pi$ so that we can apply the theorem. □

A Coisotropic submanifolds and symplectic reduction

We explain the concepts of coisotropic submanifolds and symplectic reduction following Weinstein [W].

Let M be a symplectic manifold of dimension $2n$ and $A \subset M$ be a r–codimensional submanifold. In the tangent bundle the symplectic complement is defined by

$$T_xA^{\omega} = \{\xi \in T_xM \mid \omega(\xi, \eta) = 0 \,\forall\, \eta \in T_xA\}$$

The submanifold A is called coisotropic if $T_xA^{\omega} \subset T_xA$. If $T_xA^{\omega} = T_xA$ we call A Lagrangian. In this case we have $dimA = n$. Another important example of coisotropic submanifolds are hypersurfaces as level surfaces of functions $M \to \mathbb{R}$.

T_xA^{ω} defines a r–dimensional distribution on T_xA. The distribution $T_xA^{\omega} =: \mathcal{L}_A$ is called characteristic distribution and in the case of hypersurfaces characteristic line bundle.

There is a particularly interesting class of hypersurfaces

Definition A.1 *A hypersurface $\Sigma \subset \mathbb{R}^{2n}$ is of contact type if on a neighbourhood of Σ there exists a 1–form λ such that $d\lambda = \omega$ and $\lambda \wedge \omega^{n-1}$ is a volume form on Σ.*

Σ is of restricted contact type if λ can be defined globally

Example: Starshaped domains.

We return to an arbitrary coisotropic submanifold A. We observe that for two vector fields ξ_1, ξ_2 in TA^{ω} and any vector field η in TA

$$0 = d\omega(\xi_1, \xi_2, \eta) = -\omega([\xi_1, \xi_2], \eta).$$

Hence $[\xi_1, \xi_2] \in TA^{\omega}$. By Frobenius' theorem the distribution is integrable. we denote by $L_A(x)$ the leaf through $x \in A$. We locally have a $(n-r)$–dimensional manifold M_A whose tangent space is given by T_xA/T_xA^{ω}. We call M_A reduction of M by Q and write $\pi : A \to M_A$. In fact M_A is a symplectic manifold because ω is independent on the point chosen on the leaf of the foliation:

Let ξ be a vector field with $\xi(x) \in T_xA^\omega$.

$$L_\xi\omega = d(i_\xi\omega) + i_\xi d\omega = 0.$$

Denote by ω_A the reduced form. We have for $\xi_1, \xi_2 \in T_xA$

$$\omega(\xi_1, \xi_2) = \omega_A(\pi(\xi_1), \pi(\xi_2)).$$

We next need the following

Proposition A.2 *Suppose that the reduction of M by A exists. Let $L \subset M$ be Lagrangian such that L intersects A transversally, that is $T_xA + T_xL = T_xM$ for any $x \in L \cap A$.*

Then the reduction $\pi(L \cap A)$ is an immersed Lagrangian submanifold of M_A.

Proof: First we show that $TL \cap TA^\omega = \{0\}$. Assume that $\xi \in TL \cap TA^\omega$ and $\xi \neq 0$. Then there exists $\eta \in TM$ such that $\omega(\xi, \eta) \neq 0$. Due to transversality we can write $\eta = \eta_1 + \eta_2$ with $\eta_1 \in TL$ and $\eta_2 \in TA$. But then $\omega(\xi, \eta_1) = 0$ and $\omega(\xi, \eta_2) = 0$, a contradiction.

We have for $x \in A$ that $ker(T_x\pi) = T_xA^\omega$. Hence for $\pi|_{L\cap A} : L \cap A \to M_A$ we have $ker(T_x\pi|_{L\cap A}) = \{0\}$. Hence $\pi|_{L\cap A}$ is an immersion. We have to show that $\pi(L \cap A)$ is Lagrangian. Clearly ω_A restricted to $\pi(L \cap A)$ is zero, hence $\pi(L \cap A) \subset (\pi(L \cap A))^\omega$.

For $\xi \in (T_xL \cap T_xA)^\omega$ we show that $\xi \in (T_xL \cap T_xA) + T_xA^\omega$. Since this space is mapped to $\pi(T_xL \cap T_xA)$ we then have $\pi(\xi) \in (T_xL \cap T_xA)$, hence $\pi(L \cap A) \supset (\pi(L \cap A))^\omega$.

Let $\xi \in (T_xL \cap T_xA)^\omega = T_xL^\omega + T_xA^\omega = T_xL + T_xA^\omega$. Hence $\xi = \xi_1 + \xi_2$ with $\xi_1 \in T_xL$ and $\xi_2 \in T_xA^\omega$. But $T_xA^\omega \subset T_xA$. Consequently $\xi_1 = \xi - \xi_2 \in T_xA$. Thus $\xi_1 \in T_xL \cap T_xA$ and $\xi \in (T_xL \cap T_xA) + T_xA^\omega$. $\square$

B Different Maslov indices

In his paper [A] Arnold presented two equivalent ways to define an index for loops of Lagrangian subspaces as proposed by Maslov. It uses the fact that for the set of Lagrangian planes $\Lambda(n)$ in $\mathbb{R}^{2n}$ we have $\pi_1(\Lambda(n)) = \mathbb{Z}$.

Since then many different versions to generalize this index to arbitrary paths have appeared. In their paper [CLM] Cappell, Lee and Miller presented four equivalent ways to generalize the index from [A] — two of them as direct generalizations of Arnold's ideas and two via infinite dimensional spectral flows.

They considered any index for paths of pairs of Lagrangians $(L_1(t), L_2(t))$ that satisfies the properties (i)–(iv) of proposition 4.4. (It is no problem to reformulate (i)–(iv) for pairs). In addition the index μ has to satisfy a **symplectic invariance** property: $\mu(L_1(t), L_2(t)) = \mu(\phi L_1(t), \phi L_2(t))$. We only need that for symplectic automorphisms ϕ with $\phi|_{(\mathbb{R}^n \times \{0\})} = id$ we have $\mu(\phi(\gamma)) = \mu(\gamma)$.

We now present their technique. We restrict ourselves to $a = 0$ and $b = 1$.

Consider a path $\gamma : [0, 1] \to \Lambda(n)$.

First we have to show the symplectic invariance property. We look for a homotopy $\phi(s)$ in $Sp(n)$ joining ϕ and I such that $\phi(s)|_{(\mathbb{R}^n \times \{0\})} = id$. We can write

$$\phi = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

and then find any homotopy $B(s)$ of matrices that joins B to 0. We thus found the desired homotopy.

For $\phi(s)$ and any Lagrangian L we have

$$\phi(s)(L) \cap (\mathbb{R}^n \times \{0\}) \cong L \cap (\phi(s)\,(\mathbb{R}^n \times \{0\})) = L \cap (\mathbb{R}^n \times \{0\})$$

such that the dimension of the intersection stays constant. By deformation invariance we then have $\mu(\phi(\gamma)) = \mu(\gamma)$.

We now write $\mathbb{R}^{k_0} \oplus i\mathbb{R}^{n-k_0}$ for $\mathbb{R}^{k_0} \times \{0\} \times \{0\} \times \mathbb{R}^{n-k_0} \subset \mathbb{R}^{2n}$. By symplectic invariance and symplectic additivity we can assume that $\gamma(0)$ is given in the following form: $\gamma(0) = \mathbb{R}^{k_0} \oplus i\mathbb{R}^{n-k_0}$ and $\gamma(1) = \mathbb{R}^{k_1} \oplus i\mathbb{R}^{n-k_1}$.

Now we begin with the construction. We add a tail to the ends. Consider $\gamma_k(t) = (e^{it}\mathbb{R}^k) \oplus i\mathbb{R}^{n-k}$. From [A] we know that for small $t \neq 0$ the path has trivial intersection with $\mathbb{R}^n \times \{0\}$. We then consider the path

$$\tilde{\gamma}(t) = \begin{cases} \gamma_{k_0}(t+\pi/2) \text{ for } t \in [-\pi/2, -\pi/4] \\ \gamma_{k_0}(-t) \text{ for } t \in [-\pi/4, 0] \\ \gamma(t) \text{ for } t \in [0,1] \\ \gamma_{k_1}(-1+t) \text{ for } t \in [1, 1+\pi/4] \\ \gamma_{k_1}(1+\pi/2-t) \text{ for } t \in [1+\pi/4, 1+\pi/2] \end{cases}$$

Since the added tails are traversed in both directions we have $\mu_V(\gamma) = \mu_V(\tilde{\gamma})$.

At $t = -\pi/4$ and $t = 1+\pi/4$ there is only trivial intersection with $\mathbb{R}^n \cap \{0\}$. We can deform $\tilde{\gamma}|_{[-\pi/4, 1+\pi 4]}$ to a path with which only intersects $\Lambda_1(n)$ transversally and none of the $\Lambda_k(n)$ with $k > 1$. Let us assume that it intersects $\Lambda_1(n)$ p times like $e^{it}\mathbb{R} \times i\mathbb{R}^{n-1}$ and q times like $e^{-it}\mathbb{R} \times i\mathbb{R}^{n-1}$.

We define $x = \mu(e^{it}\mathbb{R}|_{t\in[0,\pi/4]})$ and $y = \mu(e^{-it}\mathbb{R}|_{t\in[-\pi/4,0]})$. (A subscript at x and y will in the following denote a special index.) Close to the end points we can compute the Maslov indices by symplectic additivity.

Summing up we obtain $\mu(\gamma) = \mu(\tilde{\gamma}) = k_0 x + (x+y)p - (x+y)q + k_1 y$.

As proved in proposition 4.4 we have for μ_V that

$$x_V = 0 \text{ and } y_V = -1.$$

Consequently

$$\mu_V(\gamma) = -(p-q) - k_1.$$

The index of Cappell, Lee and Miller

The index μ_{CLM} of Cappell, Lee and Miller satisfies $x_{CLM} = 1$ and $y_{CLM} = 0$. Consequently for any path γ

$$\mu_{CLM}(\gamma) = k_0 + (p-q)$$

and

$$\mu_V(\gamma) = -\mu_{CLM}(\gamma) + \dim(\gamma(0) \cap (\mathbb{R}^n \times \{0\})) - \dim(\gamma(1) \cap (\mathbb{R}^n \times \{0\}))$$

For loops the index μ_{CLM} agrees with the Maslov index defined by Arnold. Consequently μ_V is the negative Maslov index for such paths.

The index of Robbin and Salamon

In [RS] Robbin and Salamon defined another Maslov index μ_{RS} via the signature of an intersection form. They count signs at starting and end point with a factor 1/2.

Consequently $x_{RS} = y_{RS} = 1/2$ and $\mu_{RS}(\gamma) = 1/2k_0 + (p - q) + 1/2k_1$. Hence

$$\begin{aligned} & \mu_V(\gamma) \\ = \ & -\mu_{RS}(\gamma) + 1/2(\dim(\gamma(0) \cap (\mathbb{R}^n \times \{0\})) - \dim(\gamma(1) \cap (\mathbb{R}^n \times \{0\})). \end{aligned}$$

References

[A] *Arnold, V.: Characteristic class entering in quantization conditions, Functional Analysis and its Applications,***1***,1–14, (1967)*

[Bol] *Bolle, P.: Une condition de contact pour les sous-varits cosotropes d'une varit symplectique, C. R. Acad. Sci. Paris Sr. I Math.,* **322***, 83–86, (1996)*

[Bor] *Born, S: Erzeugende Funktionen und symplektische Kapazitäten, diploma thesis, Heidelberg (1994)*

[CLM] *Capell, S., Lee, R. and Miller, F.: On the Maslov index, Comm. Pure and Appl. Math.,* **47***, 121–186, (1994)*

[Ce] *Cerf, J.: La stratification naturelle des espaces de fonctions différentiables réeles et le théorème de la pseudo–isotopie, Publ. Math., Inst. Hautes Étud. Sci.* **39***, 5–173 (1971)*

[Ch] *Chaperon, M.: Une idée du type géodésiques brisés pour les systèmes hamiltoniens, C. R. Acad. Sci. Paris* **298***, 293-296, (1984)*

[CZ] *Conley, C. and Zehnder, E.: Morse–type index theory for flows and periodic solutions of Hamiltonian equations, Comm. Pure and Appl. Math.,* **37***, 207–253, (1984)*

[Du] *Duistermaat, J.J.: On the Morse index in variational calculus, Advances in Math.,* **21***, 173– 195, (1976)*

[EH] *Ekeland, E. and Hofer, H.: Symplectic Topology and Hamiltonian Dynamics I, Math. Zeit.* **200***, 355-378, (1989)*

[Gr] *Gromov, M.: Pseudoholomorphic curves in symplectic manifolds, Invent. Math.,* **82***, 307–347, (1985)*

[GS] *Guillemin, V. and Sternberg, S.: Geometric asymptotics, Providence, (1977)*

[H] *Hofer, H.: On the topological properties of symplectic maps, Proceedings of the Royal Society of Edinburgh,* **115A***, 25–38, (1990)*

[HWZ1] *Hofer, H., Wysocki, C. and Zehnder, E.: Properties of holomorphic curves in symplectisations II: Embedding controls and algebraic invariants, Geometric and functional analysis,* **5**, *270–328, (1995)*

[HWZ2] *Hofer, H., Wysocki, C. and Zehnder, E.: The dynamics on a strictly convex energy surface in* $\mathbb{R}^4$, *ETH preprint (1996)*

[HZ] *Hofer, H. and Zehnder, E.: Symplectic invariants and Hamiltonian Dynamics, Birkhäuser, (1994)*

[Ho] *Hörmander, L.: Fourier integral operators I., Acta Math.,* **127**, *79–183 (1971)*

[K] *Kato, T.: Perturbation theory for linear operators, Springer, Berlin, Heidelberg, (1966)*

[M] *Moser, J.: A Fixed Point Theorem in Symplectic Geometry, Acta Math.,* **141**, *17–34, (1978)*

[R] *Rabinowitz, P.: Periodic solutions of Hamiltonian systems, Comm. Pure. and Appl. Math.,* **31**, *157–184, (1978)*

[RS] *Robbin, J. and Salamon, D.: The Maslov index for paths, Topology* **32**, *827–844, (1993)*

[SZ] *Salamon, D. and Zehnder, E.: Morse theory for periodic solutions of Hamiltonian systems and the Maslov index, Comm. Pure Appl. Math.,* **45**, *1303–1360, (1992)*

[S] *Sikorav, J. C. : Sur les immersions lagrangiennes admettant une phase génératrice globale, C.R. Acad. Sci., Paris, Sér. I,* **301**, *119–122, (1986)*

[Th] *Théret, D.: Utilisation des fonctions génératrices en géométrie symplectique globale, Thèse de Doctorat, Université Paris 7, (1996)*

[To] *Tolkien, J.R.R.: The Hobbit, George Allen & Unwin, 1937*

[Tr] *Traynor, L.: Symplectic Homology via Generating Functions, Geometric and functional analysis,* **4**, *718–748, (1994)*

[V1] *Viterbo, C.: Intersections de sous variétés lagrangiennes, fonctionelles d'action et indices des systèmes hamiltoniens, Bull. Soc. math. France* **115**, *361–390, (1987)*

[V2] *Viterbo, C.: Symplectic Topology as the Geometry of Generating Functions, Math. Ann* **292**, *685–710, (1992)*

[W] *Weinstein, A.: Lectures on Symplectic Manifolds, Providence, (1977)*